

Bohr-Sommerfeld-Heisenberg Quantization of the 2-dimensional Harmonic Oscillator

Richard Cushman and Jędrzej Śniatycki

Abstract

We perform a Bohr-Sommerfeld-Heisenberg quantization of the 2-dimensional harmonic oscillator, and obtain a reducible unitary representation of $SU(2)$. Each energy level carries an irreducible unitary representation. This leads to a decomposition of the representation of $SU(2)$, obtained by quantization of the harmonic oscillator, into direct sum of irreducible unitary representations.

Classical reduction of the energy level, corresponding to an irreducible unitary representation, gives the corresponding coadjoint orbit. However, the representation obtained by Bohr-Sommerfeld-Heisenberg quantization of the coadjoint orbit gives a unitary representation of $SU(2)$ that is the direct sum of two irreducible representations.

1 Introduction

In the framework of geometric quantization we have formulated a quantization theory based on the Bohr-Sommerfeld quantization rules [1], [9]. The aim of this theory is to provide an alternative approach to quantization of a completely integrable system that incorporates the Bohr-Sommerfeld spectrum of the defining set of commuting dynamical variables of the system. The resulting theory closely resembles Heisenberg's matrix theory and we refer to it as the Bohr-Sommerfeld-Heisenberg quantization [4].

In this paper we give a detailed treatment of Bohr-Sommerfeld-Heisenberg quantization of the 2-dimensional harmonic oscillator. We show that this approach to quantization yields all the usual results including the quantization representation of $U(2)$. We also reduce of oscillator symmetry of the

2-dimensional harmonic oscillator, quantize the reduced space, and discuss commutation of quantization and reduction in this context.

Let P be a smooth manifold with symplectic form ω . Consider a completely integrable system $(f_1, \dots, f_n, P, \omega)$, where n is equal to $\frac{1}{2} \dim P$, and f_1, \dots, f_n are Poisson commuting functions that are independent on an open dense subset U of P . If all integral curves of the Hamiltonian vector fields of f_1, \dots, f_n are closed, then the joint level sets of f_1, \dots, f_n form a singular foliation of (P, ω) by n -dimensional Lagrangian tori. The restriction of this singular foliation to the open dense subset U of P is a regular foliation of $(U, \omega|_U)$.

Each torus T of the foliation has a neighbourhood W in P such that the restriction of ω to W is exact, see [2, appendix D]. In other words, $\omega|_W = d\theta_W$. The modern version of Bohr-Sommerfeld quantization rules requires that for each generator Γ_i of the fundamental group of the n -torus T , we have

$$\int_{\Gamma_i} \theta_W = m_i h \text{ for } i = 1, \dots, n, \quad (1)$$

where m_i is an integer and h is Planck's constant. It is easy to verify that the Bohr-Sommerfeld conditions are independent of the choice of the form θ_W satisfying $d\theta_W = \omega|_W$. Intrinsically, the Bohr-Sommerfeld conditions are equivalent to the requirement that the connection on the prequantization line bundle L when restricted to T has trivial holonomy group [7].

Let S be the collection of all tori satisfying the Bohr-Sommerfeld condition. We refer to S as the Bohr-Sommerfeld set of the integrable system $(f_1, \dots, f_n, P, \omega)$. Since the curvature form of L is symplectic, it follows that the complement of S is open in P . Hence, the representation space \mathfrak{H} of geometric quantization of an integrable system consists of distribution sections of L supported on the Bohr Sommerfeld set S . Since these distributional sections are covariantly constant along the leaves of the foliation by tori, it follows that each n -torus $T \in S$ corresponds to a 1-dimensional subspace \mathfrak{H}_T of \mathfrak{H} . We choose an inner product $(\cdot | \cdot)$ on \mathfrak{H} so that the family $\{\mathfrak{H}_T \mid T \in S\}$ consists of mutually orthogonal subspaces.

In Bohr-Sommerfeld quantization, one assigns to each n -tuple of Poisson commuting constants of motion $f = (f_1, \dots, f_n)$ on P an n -tuple $(\mathbf{Q}_{f_1}, \dots, \mathbf{Q}_{f_n})$ of commuting quantum operators \mathbf{Q}_{f_k} for $1 \leq k \leq n$ such that for each n -torus $T \in S$, the corresponding 1-dimensional space \mathfrak{H}_T of the representation

space $(\mathfrak{H}, (| \rangle))$ is an eigenspace for each \mathbf{Q}_{f_k} for $1 \leq k \leq n$ with eigenvalue $f_k|_T$. For any smooth function $F \in C^\infty(\mathbb{R}^n)$, the composition $F(f_1, \dots, f_n)$ is quantizable. The operator $\mathbf{Q}_{F(f_1, \dots, f_n)}$ acts on each \mathfrak{H}_T by multiplication by $F(f_1, \dots, f_n)|_T$.

We assume that there exist global action-angle variables (A_i, φ_i) on U such that $\omega|_U = d(\sum_{i=1}^n A_i d\varphi_i)$. Therefore, we can replace θ_W by the 1-form $\sum_{i=1}^n A_i d\varphi_i$ in equation (1) and rewrite the Bohr sommerfeld conditions as

$$\int_{\Gamma_i} A_i d\varphi_i = m_i h \text{ for each } i = 1, \dots, n. \quad (2)$$

Since the actions A_i are independent of the angle variables, we can perform the integration and obtain

$$A_i = m_i \hbar \text{ for each } i = 1, \dots, n. \quad (3)$$

where $\hbar = h/2\pi$. For each multi-index $\mathbf{m} = (m_1, \dots, m_n)$, we denote by $T_{\mathbf{m}}$ the torus satisfying the Bohr-Sommerfeld conditions with integers m_1, \dots, m_n on the right hand side of equation (3).

Let S_U be the restriction of the Bohr-Sommerfeld set S to the open neighbourhood U of P on which the functions f_1, \dots, f_n are independent. Consider a subspace \mathfrak{H}_U of \mathfrak{H} given by the direct sum of 1-dimensional subspaces $\mathfrak{H}_{T_{\mathbf{m}}}$ corresponding to Bohr-Sommerfeld tori $T_{\mathbf{m}} \in S_U$. Let $\mathbf{e}_{\mathbf{m}}$ be a basis vector of $\mathfrak{H}_{T_{\mathbf{m}}}$. Each $\mathbf{e}_{\mathbf{m}}$ is a joint eigenvector of the commuting operators $(\mathbf{Q}_{A_1}, \dots, \mathbf{Q}_{A_n})$ corresponding to the eigenvalue $(m_1 \hbar, \dots, m_n \hbar)$. The vectors $(\mathbf{e}_{\mathbf{m}})$ form an orthogonal basis in \mathfrak{H}_U . Thus,

$$(\mathbf{e}_{\mathbf{m}} | \mathbf{e}_{\mathbf{m}'}) = 0 \text{ if } \mathbf{m} \neq \mathbf{m}'. \quad (4)$$

For each $i = 1, \dots, n$, introduce an operator \mathbf{a}_i on \mathfrak{H}_U such that

$$\mathbf{a}_i \mathbf{e}_{(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_n)} = \mathbf{e}_{(m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_n)}. \quad (5)$$

In other words, the operator \mathbf{a}_i shifts the joint eigenspace of $(\mathbf{Q}_{A_1}, \dots, \mathbf{Q}_{A_n})$ corresponding to the eigenvalue $(m_1 \hbar, \dots, m_n \hbar)$ to the joint eigenspace of $(\mathbf{Q}_{A_1}, \dots, \mathbf{Q}_{A_n})$ corresponding to the eigenvalue $(m_1 \hbar, \dots, m_{i-1} \hbar, (m_i - 1) \hbar, m_{i+1} \hbar, \dots, m_n \hbar)$. Let \mathbf{a}_i^\dagger be the adjoint of \mathbf{a}_i . Equations (4) and (5) yield

$$\mathbf{a}_i^\dagger \mathbf{e}_{(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_n)} = \mathbf{e}_{(m_1, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots, m_n)} \quad (6)$$

We refer to the operators \mathbf{a}_i and \mathbf{a}_i^\dagger as *shifting* operators.¹ For every $i = 1, \dots, n$, we have

$$[\mathbf{a}_i, \mathbf{Q}_{A_j}] = \hbar \mathbf{a}_i \delta_{ij}, \quad (7)$$

where δ_{ij} is Kronecker's symbol that is equal to 1 if $i = j$ and vanishes if $i \neq j$. Taking the adjoint, of the preceding equation, we get

$$[\mathbf{a}_i^\dagger, \mathbf{Q}_{A_i}] = -\hbar \mathbf{a}_i^\dagger \delta_{ij}.$$

The operators \mathbf{a}_i and \mathbf{a}_i^\dagger are well defined in the Hilbert space \mathfrak{H} . In [4], we have interpreted them as the result of quantization of appropriate functions on P . We did this as follows. We look for smooth complex-valued functions h_i on P satisfying the Poisson bracket relations

$$\{h_j, A_i\} = i\delta_{ij}h_j. \quad (8)$$

The Dirac quantization condition (9)

$$[\mathbf{Q}_f, \mathbf{Q}_h] = -i\hbar \mathbf{Q}_{\{f,h\}} \quad (9)$$

implies that we may interpret the operator \mathbf{a}_j as the quantum operator corresponding to h_j . In other words, we set $\mathbf{a}_j = \mathbf{Q}_{h_j}$. This choice is consistent with equation (8) because (7) yields

$$[\mathbf{Q}_{h_j}, \mathbf{Q}_{A_i}] = -i\hbar \mathbf{Q}_{\{h_j, A_i\}} = -i\hbar \mathbf{Q}_{i\delta_{ij}h_j} = \delta_{ij}\hbar \mathbf{Q}_{h_j}.$$

Since $\omega|_U = \sum_{i=1}^n dA_i \wedge d\varphi_i$, it follows that the Poisson bracket of $e^{i\varphi_j}$ and A_i is

$$\{e^{i\varphi_j}, A_i\} = X_{A_i} e^{i\varphi_j} = \frac{\partial}{\partial \varphi_i} e^{i\varphi_j} = i\delta_{ij} e^{i\varphi_j}. \quad (10)$$

Comparing equations (8) and (10) we see that we may make the following identification $\mathbf{a}_i = \mathbf{Q}_{e^{i\varphi_i}}$ and $\mathbf{a}_i^\dagger = \mathbf{Q}_{e^{-i\varphi_i}}$ for $i = 1, \dots, n$. Clearly, the functions $h_j = e^{i\varphi_j}$ are not uniquely defined by equation (8). We can multiply them by arbitrary functions that commute with all actions A_1, \dots, A_n . Hence, there is a choice involved. We shall use this freedom of choice to satisfy consistency requirements on the boundary of U in P .

¹In representation theory, shifting operators are called ladder operators. The corresponding operators in quantum field theory are called the creation and annihilation operators.

2 The classical theory

We now describe the symplectic geometry of the 2-dimensional harmonic oscillator and then reduce the S^1 oscillator symmetry, which is generated by its motion.

The configuration space of the 2-dimensional harmonic oscillator is \mathbb{R}^2 with coordinates $x = (x_1, x_2)$. The phase space is $T^*\mathbb{R}^2 = \mathbb{R}^4$ with coordinates $(x, y) = (x_1, x_2, y_1, y_2)$. On $T^*\mathbb{R}^2$ the canonical 1-form is $\Theta = y_1 dx_1 + y_2 dx_2 = \langle y, dx \rangle$. The symplectic form on $T^*\mathbb{R}^2$ is the closed nondegenerate 2-form $\omega = d\Theta = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$ whose matrix representation is

$$\omega = \begin{pmatrix} dx \\ dy \end{pmatrix}^T \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}. \quad (11)$$

The structure matrix $\mathcal{W}(x, y) = \left(\begin{array}{c|c} \{x_i, x_j\} & \{x_i, y_j\} \\ \hline \{y_i, x_j\} & \{y_i, y_j\} \end{array} \right)$ of the Poisson bracket $\{, \}$ on $C^\infty(T^*\mathbb{R}^2)$ is $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$.

The Hamiltonian function of the 2-dimensional harmonic oscillator is

$$E : T^*\mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto \frac{1}{2} (x_1^2 + y_1^2) + \frac{1}{2} (x_2^2 + y_2^2). \quad (12)$$

The corresponding Hamiltonian vector field $X_E = \langle X_1, \frac{\partial}{\partial x} \rangle + \langle X_2, \frac{\partial}{\partial y} \rangle$ can be computed using $-dE = X_E \lrcorner \omega = \langle X_2, dx \rangle - \langle X_1, dy \rangle$. We get

$$X_1 = \frac{\partial E}{\partial y} \quad \text{and} \quad X_2 = -\frac{\partial E}{\partial x}. \quad (13)$$

Therefore the equations of motion of the harmonic oscillator are

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -x. \quad (14)$$

The solution to the above equations is the one parameter family of transformations

$$\phi_t^E(x, y) = A(t) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\cos t) I_2 & -(\sin t) I_2 \\ (\sin t) I_2 & (\cos t) I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (15)$$

This defines an S^1 action, called the oscillator symmetry, on $T^*\mathbb{R}^2$, which is a map from \mathbb{R} to $\text{Sp}(4, \mathbb{R})$ that sends t to the 4×4 symplectic matrix $A(t)$, which is periodic of period 2π .

Since E is constant along the integral curves of X_E , the manifold

$$E^{-1}(e) = \{(x, y) \in \mathbb{R}^4 \mid x^2 + y^2 = 2e, e > 0\} = S_{\sqrt{2e}}^3, \quad (16)$$

which is a 3-sphere of radius $\sqrt{2e}$, is invariant under the flow of X_E .

The configuration space \mathbb{R}^2 is invariant under the S^1 action $S^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (t, x) \mapsto R_t x$, where R_t is the matrix $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. This map lifts to a symplectic action Ψ_t of S^1 on phase space $T^*\mathbb{R}^2$ that sends (x, y) to $\Psi_t(x, y) = (R_t x, R_t y)$. The infinitesimal generator of this action is

$$Y(x, y) = \left. \frac{d}{dt} \right|_{t=0} \Phi_t(x, y) = (-x_2, x_1, -y_2, y_1). \quad (17)$$

The vector field Y is Hamiltonian corresponding to the Hamiltonian function

$$L(x, y) = \langle y, (x_2, -x_1) \rangle = x_1 y_2 - x_2 y_1, \quad (18)$$

that is, $Y = X_L$. L is readily recognized as the angular momentum. The integral curve of the vector field X_L on \mathbb{R}^4 starting at $(x, y) = (x_1, x_2, y_1, y_2)$ is

$$s \mapsto \varphi_s^L(x, y) = \begin{pmatrix} \cos s & \sin s & 0 & 0 \\ -\sin s & \cos s & 0 & 0 \\ 0 & 0 & \cos s & \sin s \\ 0 & 0 & -\sin s & \cos s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which is periodic of period 2π . The Hamiltonian of the harmonic oscillator is an integral of X_L . The S^1 symmetry Ψ of the harmonic oscillator is called the angular momentum symmetry.

Consider the completely integrable system $(E, L, \mathbb{R}^4, \omega)$ where E (12) and L (18) are the energy and angular momentum of the 2-dimensional harmonic oscillator. The Hamiltonian vector fields

$$X_E = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial y_1} - x_2 \frac{\partial}{\partial y_2} \quad \text{and} \quad X_L = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2}$$

corresponding to the Hamiltonians E and L , respectively, define the generalized distribution

$$D : \mathbb{R}^4 \rightarrow T\mathbb{R}^4 : (x, y) \mapsto D_{(x, y)} = \text{span}\{X_E(x, y), X_L(x, y)\}. \quad (19)$$

Before we can investigate the geometry of D (19) we will need some very detailed geometric information about the 2-torus action

$$\Phi : T^2 \times \mathbb{R}^4 \rightarrow \mathbb{R}^4 : ((t_1, t_2), (x, y)) \mapsto \varphi_{t_1}^E \circ \varphi_{t_2}^L(x, y) \quad (20)$$

generated by the 2π periodic flows $\varphi_{t_1}^E$ and $\varphi_{t_2}^L$ of the vector fields X_E and X_L , respectively. Using invariant theory we find the space $V = \mathbb{R}^4/T^2$ of orbits

of the T^2 -action Φ (20) as follows. The algebra of T^2 -invariant polynomials on \mathbb{R}^4 is generated by $\sigma_1 = \frac{1}{2}(y_1^2 + x_1^2 + y_2^2 + x_2^2)$ and $\sigma_2 = x_1y_2 - x_2y_1$. Let $\sigma_3 = \frac{1}{2}(y_1^2 + x_1^2 - y_2^2 - x_2^2)$ and $\sigma_4 = x_1y_1 + x_2y_2$. Then $\sigma_1^2 = \sigma_2^2 + \sigma_3^2 + \sigma_4^2 \geq \sigma_2^2$. From the fact that $\sigma_1 \geq 0$, we get $\sigma_1 \geq |\sigma_2|$. Thus if $\sigma_1 = e \geq 0$, then $-e \leq \ell = \sigma_2 \leq e$. So the T^2 orbit space V is the semialgebraic variety $\{(e, \ell) \in \mathbb{R}^2 \mid |\ell| \leq e\}$. As a differential space V has a differential structure equal to the space $C^\infty(\mathbb{R}^4)^{T^2}$ of smooth T^2 -invariant functions on \mathbb{R}^4 , that is, the space of smooth functions in σ_1 and σ_2 . Note that the T^2 -orbit map $\pi : \mathbb{R}^4 \rightarrow V \subseteq \mathbb{R}^2 : (x, y) \mapsto (\sigma_1(x, y), \sigma_2(x, y))$ is just the energy momentum mapping

$$\mathcal{EM} : \mathbb{R}^4 \rightarrow \mathbb{R}^2 : (x, y) \mapsto (E(x, y), L(x, y))$$

of the 2-dimensional harmonic oscillator. Abstractly, the Whitney stratification of V is given by

$$V_2 = V \setminus V_{\text{sing}} \supseteq V_1 = V_{\text{sing}} \setminus (V_{\text{sing}})_{\text{sing}} \supseteq V_0 = (V_{\text{sing}})_{\text{sing}} \supseteq \emptyset,$$

where W_{sing} is the semialgebraic variety of singular points of the semialgebraic variety W . In our situation $V_2 = \{(e, \ell) \in \mathbb{R}^2 \mid |\ell| < e\}$, $V_1 = \{(e, \ell) \in \mathbb{R}^2 \mid 0 < |\ell| = e\}$, and $V_0 = \{(e, \ell) \in \mathbb{R}^2 \mid e = \ell = 0\}$. Then $\{V_j, j = 0, 1, 2\}$ is the Whitney stratification of V .

For $j = 0, 1, 2$ let $U_j = \mathcal{EM}^{-1}(V_j)$. It is straightforward to see that the isotropy group $T_{(x,y)}^2$ at (x, y) of the T^2 -action Φ (20) is

$$T_{(x,y)}^2 = \begin{cases} \{e\}, & \text{if } (x, y) \in U_2 \\ S^1 & \text{if } (x, y) \in U_1 \\ T^2, & \text{if } (x, y) \in U_0. \end{cases}$$

Thus $\{U_j, j = 0, 1, 2\}$ is the stratification of \mathbb{R}^4 by orbit type, which is equal to T^2 -symmetry type where all the isotropy groups are equal, since T^2 is abelian. Because V_j is the image of U_j under the energy momentum map, we see that U_j is the subset of \mathbb{R}^4 where the rank of $D\mathcal{EM}$ is j . Since

$$\omega^\sharp(x, y) : T_{(x,y)}\mathbb{R}^4 \rightarrow T_{(x,y)}^*\mathbb{R}^4 : X_E(x, y) \mapsto dE(x, y),$$

is bijective, it follows that

$$\dim D_{(x,y)} = \dim \text{span}\{dE(x, y), dL(x, y)\} = \text{rank } D\mathcal{EM}(x, y). \quad (21)$$

Therefore U_j is the union of j -tori for $j = 0, 1, 2$ each of which is an energy-momentum level set of \mathcal{EM} .

We turn to investigating the generalized distribution D (19). Since $\omega(X_E, X_L) = \{E, L\}$, we see that for every $(x, y) \in \mathbb{R}^4$ we have $\omega(x, y)D_{(x,y)} = 0$. Thus every subspace $D_{(x,y)}$ in the distribution D is an ω -isotropic subspace of the symplectic vector space $(T_{(x,y)}\mathbb{R}^4, \omega(x, y))$. If $(x, y) \in U_j$ then $\dim D_{(x,y)} = j$. So if $(x, y) \in U_2$, then $D_{(x,y)}$ is a Lagrangian subspace of $(T_{(x,y)}\mathbb{R}^4, \omega(x, y))$. If $u_{(x,y)}, v_{(x,y)} \in D_{(x,y)}$, then thinking of $u_{(x,y)}$ and $v_{(x,y)}$ as vector fields on $T\mathbb{R}^4$ we obtain $[u_{(x,y)}, v_{(x,y)}] = 0$, because their flows commute. Thus D is an involutive generalized distribution, which is called the energy-momentum polarization of the symplectic manifold (\mathbb{R}^4, ω) . The leaf (= integral manifold) $L_{(x,y)}$ of the distribution D through the point $(x, y) \in \mathbb{R}^4$ is diffeomorphic to $T^2/T_{(x,y)}^2$. In particular we have

$$L_{(x,y)} = \begin{cases} T^2, & \text{if } (x, y) \in U_2 \\ T^1 = S^1, & \text{if } (x, y) \in U_1 \\ T^0 = \text{pt}, & \text{if } (x, y) \in U_0. \end{cases}$$

Thus D is a generalized toric distribution. Note that the leaves of D , each of which is an energy-momentum level set of \mathcal{EM} , foliate the strata U_j for $j = 0, 1, 2$. Thus the space \mathbb{R}^4/D of leaves of D is the T^2 orbit space V .

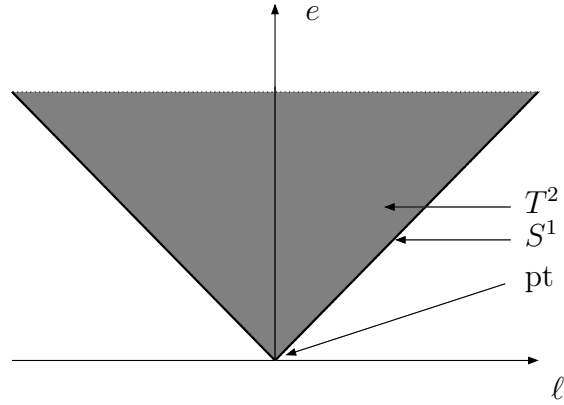


Figure 1. The bifurcation diagram of the energy momentum map.

We collect all the geometric information about the leaves of the generalized distribution D in the bifurcation diagram of the energy momentum

map of the 2-dimensional harmonic oscillator above. The range of this map is indicated by the shaded region.

We now find action-angle coordinates for the 2-dimensional harmonic oscillator. Let

$$\begin{aligned} x_1 &= \frac{-1}{\sqrt{2}}(r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) & y_1 &= \frac{1}{\sqrt{2}}(r_1 \sin \vartheta_1 + r_2 \sin \vartheta_2) \\ x_2 &= \frac{1}{\sqrt{2}}(-r_1 \sin \vartheta_1 + r_2 \sin \vartheta_2) & y_2 &= \frac{1}{\sqrt{2}}(-r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2). \end{aligned} \quad (22)$$

A computation shows that $E(r, \vartheta) = \frac{1}{2}(r_1^2 + r_2^2)$ and $L(r, \vartheta) = \frac{1}{2}(r_1^2 - r_2^2)$ and that the change of coordinates (22) pulls back the symplectic form $\omega = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$ to the symplectic form $\Omega = d(\frac{1}{2} r_1^2) \wedge d\vartheta_1 + d(\frac{1}{2} r_2^2) \wedge d\vartheta_2$. Let

$$A_1 = \frac{1}{2} r_1^2 = \frac{1}{2} (E(r, \vartheta) + L(r, \vartheta)) \geq 0$$

and

$$A_2 = \frac{1}{2} r_2^2 = \frac{1}{2} (E(r, \vartheta) - L(r, \vartheta)) \geq 0.$$

Then $(A_1, A_2, \vartheta_1, \vartheta_2)$ with $A_1 > 0$, and $A_2 > 0$ and symplectic form $\Omega = dA_1 \wedge d\vartheta_1 + dA_2 \wedge d\vartheta_2$ are real analytic action-angle coordinates for the 2-dimensional harmonic oscillator. These coordinates extend real analytically to the closed domain $A_1 \geq 0$ and $A_2 \geq 0$.

Let \mathbb{R}^4 have coordinates $(\xi, \eta) = (r_1 \cos \vartheta_1, r_1 \sin \vartheta_1, r_2 \cos \vartheta_2, r_2 \sin \vartheta_2)$ with symplectic form $d\eta_1 \wedge d\xi_1 + d\eta_2 \wedge d\xi_2 = r_1 dr_1 \wedge d\vartheta_1 + r_2 dr_2 \wedge d\vartheta_2$. For $j = 1, 2$ let $z_j = \xi_j + i\eta_j = r_j e^{i\vartheta_j}$. Then \mathbb{R}^4 becomes \mathbb{C}^2 with coordinates (z_1, z_2) . The flow φ_t^E of the vector field X_E becomes the complex S^1 -action

$$\varphi : S^1 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2 : (t, (z_1, z_2)) \mapsto (e^{it} z_1, e^{it} z_2).$$

The algebra of S^1 -invariant polynomials on \mathbb{R}^4 is generated by

$$\begin{aligned} \pi_1 &= \operatorname{Re} \bar{z}_1 z_2 & \pi_2 &= \operatorname{Im} \bar{z}_1 z_2 & \pi_3 &= \frac{1}{2} (z_1 \bar{z}_1 - z_2 \bar{z}_2) & \pi_4 &= \frac{1}{2} (z_1 \bar{z}_1 + z_2 \bar{z}_2) \\ &= \xi_1 \xi_2 + \eta_1 \eta_2 & &= \xi_1 \eta_2 - \xi_2 \eta_1 & &= \frac{1}{2} (\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2) & &= \frac{1}{2} (\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2) \end{aligned}$$

subject to the relation

$$\pi_1^2 + \pi_2^2 = |\bar{z}_1 z_2|^2 = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = (\pi_4 + \pi_3)(\pi_4 - \pi_3) = \pi_4^2 - \pi_3^2, \quad \pi_4 \geq 0.$$

Note that L expressed in the (ξ, η) coordinates is π_3 .

Using the Poisson bracket on $C^\infty(\mathbb{R}^4)$ we get the structure matrix $W(\pi) = (\{\pi_i, \pi_j\})$ of the Poisson bracket on \mathbb{R}^4 with coordinates (π_1, \dots, π_4) , namely

$\{\pi_i, \pi_j\}$	π_1	π_2	π_3	π_4
π_1	0	$2\pi_3$	$-2\pi_2$	0
π_2	$-2\pi_3$	0	$2\pi_1$	0
π_3	$2\pi_2$	$-2\pi_1$	0	0
π_4	0	0	0	0

Table 1. The Poisson bracket on $C^\infty(\mathbb{R}^4)$.

3 The Hopf fibration and reduction

In this section we discuss the Hopf fibration, which is the reduction map of the harmonic oscillator symmetry of the 2-dimensional harmonic oscillator.

Let $S^3_{\sqrt{2e}} = \{(\xi, \eta) \in \mathbb{R}^4 \mid \xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2 = 2e\}$ be the 3-sphere of radius $\sqrt{2e}$ and let $S_e^2 = \{\pi \in \mathbb{R}^3 \mid \pi_1^2 + \pi_2^2 + \pi_3^2 = e^2\}$ be the 2-sphere of radius e . The map

$$\rho : S^3_{\sqrt{2e}} \subseteq \mathbb{R}^4 \rightarrow S_e^2 \subseteq \mathbb{R}^3 : \zeta = (\xi, \eta) \mapsto \pi = (\pi_1(\zeta), \pi_2(\zeta), \pi_3(\zeta)) \quad (23)$$

is called the Hopf fibration. Let $\pi \in S_e^2$. Then $\rho^{-1}(\pi)$ is a great circle on $S^3_{\sqrt{2e}}$. We prove this as follows.

CASE 1. $\pi \in S_e^2 \setminus \{(0, 0, -e)\}$. Suppose $(\xi, \eta) \in \rho^{-1}(\pi)$. Since $\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2 = 2e$ and $\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2 = 2\pi_3$ it follows that $\xi_1^2 + \eta_1^2 = e + \pi_3 > 0$. Therefore we may solve the linear equations

$$\begin{pmatrix} \xi_1 & \eta_1 \\ -\eta_1 & \xi_1 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \quad (24)$$

to obtain

$$\begin{aligned} \pi_1 \xi_1 - \pi_2 \eta_1 - (e + \pi_3) \xi_2 &= 0 \\ \pi_2 \xi_1 + \pi_1 \eta_1 - (e + \pi_3) \eta_2 &= 0. \end{aligned}$$

The above equations define a 2-plane Π^π in \mathbb{R}^4 , since $\begin{pmatrix} \pi_1 & -\pi_2 & -(e+\pi_3) & 0 \\ \pi_2 & \pi_1 & 0 & -(e+\pi_3) \end{pmatrix}$ has rank 2. Hence $\rho^{-1}(\pi) \subseteq \Pi^\pi \cap S^3_{\sqrt{2e}}$. Reversing the argument shows that $\Pi^\pi \cap S^3_{\sqrt{2e}} \subseteq \rho^{-1}(\pi)$. Thus $\rho^{-1}(\pi)$ is a great circle.

CASE 2. $\pi = (0, 0, -e)$. Then $\xi_1^2 + \eta_1^2 = 0$ which implies $\xi_1 = \eta_1 = 0$. Thus

$$\rho^{-1}(\pi) = \{(0, \xi_2, 0, \eta_2) \in \mathbb{R}^4 \mid \xi_2^2 + \eta_2^2 = 2e\}, \quad (25)$$

which is a circle. Because $\rho^{-1}(\pi) = S_{\sqrt{2e}}^3 \cap \{\xi_1 = \eta_1 = 0\}$, it follows that $\rho^{-1}(\pi)$ is a great circle.

Each fiber of the Hopf fibration is an integral curve of the harmonic oscillator vector field X_E of energy e . Thus the space $E^{-1}(e)/S^1$ of orbits of the harmonic oscillator of energy e is S_e^2 . In other words, the Hopf fibration ρ (23) is the reduction map of the S^1 symmetry of the harmonic oscillator generated by the vector field X_E .

Consider a smooth function $K : \mathbb{R}^4 \rightarrow \mathbb{R}$, which is invariant under the flow of X_E . Then K is an integral of X_E , that is, $\mathcal{L}_{X_E} K = 0$. Moreover, there is a smooth function $\tilde{K} : \mathbb{R}^4 \rightarrow \mathbb{R}$ such that $K(\xi, \eta) = \tilde{K}(\pi_1(\xi, \eta), \pi_2(\xi, \eta), \pi_3(\xi, \eta), \pi_4(\xi, \eta))$.

Since $\{\pi_j, \pi_4\} = 0$ for $j = 1, \dots, 4$, we obtain

$$\dot{\pi}_j = \{\pi_j, \tilde{K}\} = \sum_{k=1}^3 \{\pi_j, \pi_k\} \frac{\partial \tilde{K}}{\partial \pi_k} = 2 \sum_{k=1}^3 \sum_{l=1}^3 \varepsilon_{jkl} \frac{\partial \tilde{K}}{\partial \pi_k} \pi_l = 2(\nabla \tilde{K} \times \pi)_j$$

for $j = 1, 2, 3$ and $\dot{\pi}_4 = 0$.

Restrict the function \tilde{K} to $E^{-1}(e)$ and define $\tilde{K}_e(\pi_1, \pi_2, \pi_3) = \tilde{K}(\pi_1, \pi_2, \pi_3, e)$. Set $\pi = (\pi_1, \pi_2, \pi_3) \in \mathbb{R}^3$. Then $\dot{\pi} = 2(\nabla \tilde{K}_e \times \pi)$ is satisfied by integral curves of a vector field X on \mathbb{R}^3 defined by

$$X(\pi) = 2(\nabla \tilde{K}_e \times \pi). \quad (26)$$

The 2-sphere S_e^2 is invariant under the flow of the vector field X , because

$$\mathcal{L}_X \langle \pi, \pi \rangle = 2\langle \pi, \dot{\pi} \rangle = 4\langle \pi, \nabla \tilde{K}_e(\pi) \times \pi \rangle = 0.$$

The structure matrix of the Poisson bracket $\{, \}$ on $C^\infty(\mathbb{R}^3)$ is

$$W(\pi) = (\{\pi_j, \pi_k\}) = -2 \begin{pmatrix} 0 & -\pi_3 & \pi_2 \\ \pi_3 & 0 & -\pi_1 \\ -\pi_2 & \pi_1 & 0 \end{pmatrix} \quad (27)$$

Since $\ker W(\pi) = \text{span}\{\pi\}$ and $T_\pi S_e^2 = \text{span}\{\pi\}^\perp$, the matrix $W(\pi)|_{T_\pi S_e^2}$ is invertible. On S_e^2 define a symplectic form $\omega_e(\pi)(u, v) = \langle (W(\pi)^T)^{-1}u, v \rangle$, where $u, v \in T_\pi S_e^2$. Let $y \in T_\pi S_e^2$. Since $W(\pi)^T y = 2\pi \times y = z$ we get

$$\begin{aligned}\pi \times z &= \pi \times (2\pi \times y) = 2\pi \times (\pi \times y) \\ &= 2(\pi \langle \pi, y \rangle - y \langle \pi, \pi \rangle) = -2y \langle \pi, \pi \rangle = -2e^2 y,\end{aligned}$$

which implies $y = (W(\pi)^T)^{-1}z = -\frac{1}{2e^2}\pi \times z$. Therefore

$$\omega_e(\pi)(u, v) = -\frac{1}{2e^2} \langle \pi \times u, v \rangle = -\frac{1}{2e^2} \langle \pi, u \times v \rangle. \quad (28)$$

The vector field X (26) on (S_e^2, ω_e) is Hamiltonian with Hamiltonian function \tilde{K}_e , because

$$\begin{aligned}\omega_e(\pi)(X(\pi), u) &= -\frac{1}{e^2} \langle \pi, (\nabla \tilde{K}_e \times \pi) \times u \rangle = -\frac{1}{e^2} \langle \pi \times (\nabla \tilde{K}_e \times \pi), u \rangle \\ &= -\frac{1}{e^2} \langle \nabla \tilde{K}_e \langle \pi, \pi \rangle - \pi \langle \nabla \tilde{K}_e, \pi \rangle, u \rangle = -\langle \nabla \tilde{K}_e, u \rangle \\ &= -d\tilde{K}_e(\pi)u,\end{aligned}$$

where $u, v \in T_w S_e^2$.

Since L is a function which is invariant under the flow φ_t^E of X_E on $E^{-1}(e)$, it induces the function $\tilde{\pi}_3 = \pi_3|_{S_e^2}$ on the reduced phase space S_e^2 . We look at the completely integrable reduced system $(\tilde{\pi}_3, S_e^2, \omega_e)$.

The flow of the reduced vector field $X_{\tilde{\pi}_3}(\pi) = 2(e_3 \times \pi)$ on $S_e^2 \subseteq \mathbb{R}^3$ is

$$\varphi_t^{\tilde{\pi}_3}(\pi) = e^{2tE_3}\pi = \begin{pmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & 1 \end{pmatrix} \pi,$$

where $\pi \in S_e^2$ and $E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that $\varphi_t^{\tilde{\pi}_3}$ is periodic of period π . Therefore an integral curve of $X_{\tilde{\pi}_3}$ on S_e^2 is either a circle or a point. For $|\ell| < e$ we have $\pi_3^{-1}(\ell)$ is the circle $\{(\pi_1, \pi_2, \ell) \in S_e^2 \mid \pi_1^2 + \pi_2^2 = e^2 - \ell^2\}$; while when $\ell = \pm e$, we see that $\pi^{-1}(\ell)$ is the point $(0, 0, \pm e)$.

Introduce spherical coordinates

$$\pi_1 = e \sin \theta \cos 2\psi, \quad \pi_2 = e \sin \theta \sin 2\psi, \quad \text{and} \quad \pi_3 = e \cos \theta$$

with $0 \leq \theta \leq \pi$ and $0 \leq \psi \leq \pi$. Therefore the reduced symplectic form on S_e^2 is

$$\omega_e = -\frac{1}{e}(e^2 \sin \theta) d\theta \wedge d\psi,$$

that is, $\omega_e = \frac{1}{2e} \text{vol}_{S_e^2}$. Here $\text{vol}_{S_e^2}$ is the standard volume 2-form $-\frac{1}{e} \langle \pi, u \times v \rangle$ on S_e^2 , where $\pi \in S_e^2$ and $u, v \in T_\pi S_e^2$. Note that $\int_{S_e^2} \text{vol}_{S_e^2} = 4\pi e^2$.

Following our convention, for $i = 1, 2, 3$, we denote by $\tilde{\pi}_i$ the push-forward to S_e^2 of the invariant function π_i on \mathbb{R}^3 . We now explain why the functions $\tilde{\pi}_i$ satisfy commutation relations for $\mathfrak{su}(2)$. Consider the Poisson algebra $\mathcal{A} = (C^\infty(\mathbb{R}^3), \{ , \}_{\mathbb{R}^3}, \cdot)$, where \mathbb{R}^3 has coordinates (π_1, π_2, π_3) , the Poisson bracket $\{ , \}_{\mathbb{R}^3}$ has structure matrix $W(\pi) = (\{\pi_j, \pi_k\}) = \begin{pmatrix} 0 & 2\pi_3 & -2\pi_2 \\ -2\pi_3 & 0 & 2\pi_1 \\ 2\pi_2 & -2\pi_1 & 0 \end{pmatrix}$, and \cdot is pointwise multiplication of functions. Since $C = \pi_1^2 + \pi_2^2 + \pi_3^2$ is a Casimir of \mathcal{A} , that is, $\{C, \pi_i\}_{\mathbb{R}^3} = 0$ for $i = 1, 2, 3$, it generates a Poisson ideal \mathcal{I} of \mathcal{A} . Observe that $\mathcal{I} = \{f \in C^\infty(\mathbb{R}^3) \mid f|_{S_e^2} = 0\}$, where $S_e^2 = C^{-1}(e)$. Therefore $\mathcal{B} = (C^\infty(S_e^2) = C^\infty(\mathbb{R}^3)/\mathcal{I}, \{ , \}_{S_e^2}, \cdot)$ is a Poisson algebra, where $\{\bar{f}, \bar{g}\}_{S_e^2} = \{f, g\}_{\mathbb{R}^3}|_{S_e^2}$ and $f \mapsto \bar{f}$ is the natural projection map of $C^\infty(\mathbb{R}^3)$ onto $C^\infty(S_e^2)$. Note that $\bar{f} = f|_{S_e^2}$, because every smooth function on \mathbb{R}^3 , which vanishes on S_e^2 , is a multiple of C by a smooth function on \mathbb{R}^3 . Consequently, the structure matrix \mathcal{W} of the Poisson bracket $\{ , \}_{S_e^2}$ is $\begin{pmatrix} 0 & 2\tilde{\pi}_3 & -2\tilde{\pi}_2 \\ -2\tilde{\pi}_3 & 0 & 2\tilde{\pi}_1 \\ 2\tilde{\pi}_2 & -2\tilde{\pi}_1 & 0 \end{pmatrix}$, where $\tilde{\pi}_i = \pi_i|_{S_e^2}$. In other words, $\{\tilde{\pi}_i, \tilde{\pi}_j\}_{S_e^2} = 2 \sum_{k=1}^3 \varepsilon_{ijk} \tilde{\pi}_k$, which are the bracket relations for $\mathfrak{su}(2)$, as desired.

To see why the reduced phase space S_e^2 is symplectomorphic to a coadjoint orbit of $\text{SU}(2)$ we refer the reader to the appendix.

The Hamiltonian vector field $X_{\tilde{\pi}_3}$ is $\frac{\partial}{\partial \psi}$, because

$$X_{\tilde{\pi}_3} \lrcorner \omega_e = e \sin \theta \, d\theta = -d(e \cos \theta) = -d\tilde{\pi}_3 = -da.$$

Note that the flow of $X_{\tilde{\pi}_3}$ on S_e^2 is periodic of period π . On $S_e^2 \setminus \{(0, 0, \pm e)\}$ we have

$$da \wedge d\psi = d\tilde{\pi}_3 \wedge d\psi = d(e \cos \theta) \wedge d\psi = e \sin \theta \, d\psi \wedge d\theta = \omega_e.$$

Thus $(\tilde{\pi}_3, \psi)$ are real analytic action-angle coordinates on $S_e^2 \setminus \{(0, 0, \pm e)\}$.

For later purposes we now show the inverse image under the harmonic oscillator reduction map ρ (23) of the reduced level set $\tilde{\pi}_3^{-1}(\ell)$, which is the circle $\{(\pi_1, \pi_2, \ell) \in S_e^2 \mid \pi_1^2 + \pi_2^2 = e^2 - \ell^2\}$, when $|\ell| < e$, or the point $(0, 0, \text{sgn } \ell)$, when $|\ell| = e$, is the 2-torus T_{a_1, a_2}^2 labeled by the actions $A_1 = a_1 = \frac{1}{2}(e + \ell)$ and $A_2 = a_2 = \frac{1}{2}(e - \ell)$, when $|\ell| < e$, or the 1-torus $T_{a_1, 0}^2$, where $a_1 = 2e$ when $\ell = e$ or the 1-torus T_{0, a_2}^2 , where $a_2 = 2e$ when $-\ell = e$. Suppose

that $|\ell| < e$. Because $\xi_1^2 + \xi_2^2 = e + \ell > 0$ we can solve the linear equations $\begin{pmatrix} \xi_1 & \eta_1 \\ -\eta_1 & \xi_1 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$ to obtain $\xi_2 = \frac{1}{e+\ell}(\xi_1\pi_1 - \eta_1\pi_2)$ and $\eta_2 = \frac{1}{e+\ell}(\eta_1\pi_1 + \xi_1\pi_2)$. Thus $\rho^{-1}(\tilde{\pi}_3^{-1}(\ell))$ is equal to

$$\left\{ \left(\xi_1, \frac{1}{e+\ell}(\xi_1\pi_1 - \eta_1\pi_2), \eta_1, \frac{1}{e+\ell}(\eta_1\pi_1 + \xi_1\pi_2) \right) \in \mathbb{R}^4 \mid \pi_1^2 + \pi_2^2 = e^2 - \ell^2 \text{ \& } \xi_1^2 + \xi_2^2 = e + \ell \right\}.$$

Clearly $\rho^{-1}(\tilde{\pi}_3^{-1}(\ell))$ is contained in $A^{-1}(a_1) = A_1^{-1}(\frac{1}{2}(e + \ell))$. Since

$$\xi_2^2 + \eta_2^2 = \frac{1}{(e+\ell)^2}(\xi_1^2 + \eta_1^2)(\pi_1^2 + \pi_2^2) = \frac{1}{(e+\ell)^2}(e + \ell)(e^2 - \ell^2) = e - \ell,$$

it follows that $\rho^{-1}(\tilde{\pi}_3^{-1}(\ell))$ is contained in $A^{-1}(a_2) = A^{-1}(\frac{1}{2}(e - \ell))$. Therefore $\rho^{-1}(\tilde{\pi}_3^{-1}(\ell)) = A^{-1}(a_1) \cap A^{-1}(a_2) = T_{a_1, a_2}^2$. The first equality holds since $\rho^{-1}(\tilde{\pi}_3^{-1}(\ell))$ is diffeomorphic to a 2-torus. Suppose that $|\ell| = e$, then $\begin{cases} \xi_1^2 + \eta_1^2 = 0, & \text{if } -\ell = e \\ \xi_2^2 + \eta_2^2 = 0, & \text{if } \ell = e \end{cases}$. So $\rho^{-1}(\tilde{\pi}_3^{-1}(\ell))$ is the circle $\begin{cases} \xi_1^2 + \eta_1^2 = 2e, & \text{if } \ell = e \\ \xi_2^2 + \eta_2^2 = 2e, & \text{if } -\ell = e \end{cases}$, that is, the 1-torus $T_{a_1=e, 0}^2$ if $\ell = e$, or the 1-torus $T_{0, a_2=e}^2$ if $-\ell = e$.

4 Quantization of the 2d harmonic oscillator

Following the general procedure outlined in the introduction and detailed in [4], in this section we describe the Bohr-Sommerfeld-Heisenberg quantization of the 2-dimensional harmonic oscillator in the energy-momentum representation. In §2 we have shown that action-angle coordinates are $(A_1, A_2, \vartheta_1, \vartheta_2)$, where

$$A_1 = \frac{1}{2}(E + L) \geq 0 \text{ and } A_2 = \frac{1}{2}(E - L) \geq 0. \quad (29)$$

Bohr Sommerfeld conditions applied to these action-angle variables are

$$\oint A_1 d\vartheta_1 = mh \text{ and } \oint A_2 d\vartheta_2 = nh, \quad (30)$$

where m and n are integers, and h is Planck's constant. Since the actions are independent of the angles, we can integrate equation (30) to get

$$A_1 = m\hbar \text{ and } A_2 = n\hbar,$$

where \hbar is Planck's constant divided by 2π . According to Bohr and Sommerfeld, equations (30) give the joint spectrum $\{(m\hbar, n\hbar)\}$ of the quantum operators \mathbf{Q}_{A_1} and \mathbf{Q}_{A_2} corresponding to the dynamical variables A_1 and A_2 ,

respectively. Since the actions A_1 and A_2 are nonnegative, the joint spectrum of the quantum operators $(\mathbf{Q}_{A_1}, \mathbf{Q}_{A_2})$ is the quarter lattice $\{(m\hbar, n\hbar) \in \hbar(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0})\}$.

Let \mathfrak{H} be the Hilbert space of states of the quantized harmonic oscillator. We assume that the family $\{e_{m,n} \in \mathfrak{H} \mid m \geq 0, n \geq 0\}$ of vectors is an orthogonal basis in \mathfrak{H} . For each basis vector $e_{m,n}$ of \mathfrak{H} , the 1-dimensional subspace of \mathfrak{H} spanned by $e_{m,n}$ corresponds to a 2-torus labeled by the actions (A_1, A_2) whose values are $\hbar(m, n)$.

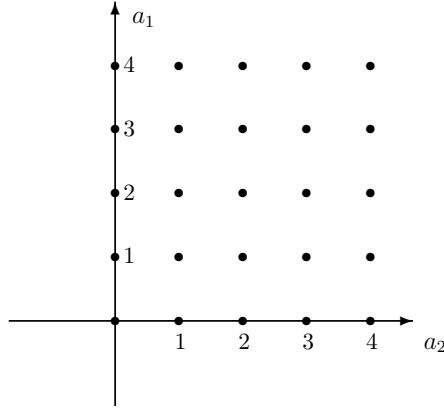


Figure 2. The quantum lattice for the joint spectrum of the quantum operators $(\mathbf{Q}_{A_1}, \mathbf{Q}_{A_2})$ of the 2-dimensional harmonic oscillator. The axes are in units of \hbar .

For each (m, n) in the first quadrant of \mathbb{Z}^2 , we note that the vector $e_{m,n}$ in \mathfrak{H} satisfies

$$\mathbf{Q}_E e_{m,n} = \mathbf{Q}_{A_1} e_{m,n} + \mathbf{Q}_{A_2} e_{m,n} = (m + n)\hbar e_{m,n}$$

and

$$\mathbf{Q}_L e_{m,n} = \mathbf{Q}_{A_1} e_{m,n} - \mathbf{Q}_{A_2} e_{m,n} = (m - n)\hbar e_{m,n}.$$

Then $((m + n)\hbar, (m - n)\hbar)$ with $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is the joint spectrum of the quantum operators $(\mathbf{Q}_E, \mathbf{Q}_L)$.

Since \mathbf{Q}_E and \mathbf{Q}_L commute, they span a 2-dimensional abelian Lie algebra \mathfrak{t}^2 . For every nonnegative integer N let \mathfrak{H}_N be the subspace of \mathfrak{H} spanned by

the vectors e_{mn} where $m + n = N$. Then $\mathfrak{H} = \bigoplus_{N \geq 0} \mathfrak{H}_N$ is an orthogonal direct sum decomposition of \mathfrak{H} into $N + 1$ -dimensional \mathfrak{t}^2 -invariant subspaces. Because every basis vector of \mathfrak{H}_N is an eigenvector of \mathbf{Q}_L , this representation of \mathfrak{t}^2 is reducible.

Following the general theory outlined in the introduction, for $j = 1, 2$ define the shifting operators \mathbf{a}_j and $(\mathbf{a}_j)^\dagger$ by

$$\mathbf{a}_1 e_{m,n} = e_{m-1,n}, \quad (\mathbf{a}_1)^\dagger e_{m,n} = e_{m+1,n}, \quad \mathbf{a}_2 e_{m,n} = e_{m,n-1}, \quad (\mathbf{a}_2)^\dagger e_{m,n} = e_{m,n+1}.$$

These operators correspond to quantum operators $\mathbf{a}_j = \mathbf{Q}_{e^{i\vartheta_j}}$ and $(\mathbf{a}_j)^\dagger = \mathbf{Q}_{e^{-i\vartheta_j}}$. However, the functions $e^{\pm i\vartheta_j}$ do not extend smoothly to the origin $(0, 0)$ in $\mathbb{R}^2 \subseteq \mathbb{R}^4$ with coordinates (ξ_j, η_j) . On the other hand, the complex conjugate coordinate functions

$$z_j = \xi_j + i\eta_j = r_j e^{i\vartheta_j} \quad \text{and} \quad \bar{z}_j = \xi_j - i\eta_j = r_j e^{-i\vartheta_j} \quad \text{for } j = 1, 2,$$

are smooth on all of \mathbb{R}^4 . Moreover, they satisfy the required Poisson bracket relations: $\{z_j, z_k\} = \{\bar{z}_j, \bar{z}_k\} = 0$ and $\{z_j, \bar{z}_k\} = -2i\delta_{jk}$. Therefore, we may introduce new quantum operators \mathbf{Q}_{z_j} and $\mathbf{Q}_{\bar{z}_j}$, where

$$\begin{aligned} \mathbf{Q}_{z_1} e_{m,n} &= \begin{cases} \sqrt{2m\hbar} e_{m-1,n}, & \text{when } m \geq 1 \\ 0, & \text{when } m = 0 \end{cases} \\ \mathbf{Q}_{z_1}^\dagger e_{m,n} &= \mathbf{Q}_{\bar{z}_1} e_{m,n} = \sqrt{2(m+1)\hbar} e_{m+1,n} \\ \mathbf{Q}_{z_2} e_{m,n} &= \begin{cases} \sqrt{2n\hbar} e_{m,n-1}, & \text{when } n \geq 1 \\ 0, & \text{when } n = 0 \end{cases} \\ \mathbf{Q}_{z_2}^\dagger e_{m,n} &= \mathbf{Q}_{\bar{z}_2} e_{m,n} = \sqrt{2(n+1)\hbar} e_{m,n+1}. \end{aligned} \tag{31}$$

Now consider polynomials $\pi_1 = \text{Re } \bar{z}_1 z_2$, $\pi_2 = \text{Im } \bar{z}_1 z_2$, π_2 and $\pi_3 = \frac{1}{2}(\bar{z}_1 z_1 - \bar{z}_2 z_2) = L$, which are invariant under the S^1 action $(t, z) \mapsto e^{it} z$, generated by the flow of the Hamiltonian vector field X_E . Look at the corresponding quantum operators \mathbf{Q}_{π_1} , \mathbf{Q}_{π_2} , and \mathbf{Q}_{π_3} . Then

$$\begin{aligned} \mathbf{Q}_{2\pi_1} e_{m,n} &= \mathbf{Q}_{z_1 \bar{z}_2} e_{m,n} + \mathbf{Q}_{\bar{z}_1 z_2} e_{m,n} = \mathbf{Q}_{z_1} \mathbf{Q}_{\bar{z}_2} e_{m,n} + \mathbf{Q}_{\bar{z}_1} \mathbf{Q}_{z_2} e_{m,n} \\ &= \mathbf{Q}_{z_1} (\sqrt{2(n+1)\hbar} e_{m,n+1}) + \mathbf{Q}_{\bar{z}_1} (\sqrt{2(m+1)\hbar} e_{m,n-1}) \\ &= 2\hbar (\sqrt{m(n+1)} e_{m-1,n+1} + \sqrt{(m+1)n} e_{m+1,n-1}), \end{aligned}$$

Similarly, $\mathbf{Q}_{2i\pi_2} e_{m,n} = 2\hbar(\sqrt{m(n+1)} e_{m-1,n+1} - \sqrt{m+1,n} e_{m+1,n-1})$ and $\mathbf{Q}_{2\pi_3} e_{m,n} = 2(m-n)\hbar e_{m,n}$. The following calculation shows that the quantum operator \mathbf{Q}_{π_1} on the Hilbert space $(\mathfrak{H}, (\cdot|\cdot))$ is self adjoint.

$$\begin{aligned} \mathbf{Q}_{2\pi_1}^\dagger e_{m,n} &= \mathbf{Q}_{z_1\bar{z}_2}^\dagger e_{m,n} + \mathbf{Q}_{\bar{z}_1 z_2}^\dagger e_{m,n} = \mathbf{Q}_{\bar{z}_2}^\dagger \mathbf{Q}_{z_1}^\dagger e_{m,n} + \mathbf{Q}_{z_2}^\dagger \mathbf{Q}_{\bar{z}_1}^\dagger e_{m,n} \\ &= \mathbf{Q}_{\bar{z}_2}^\dagger \mathbf{Q}_{\bar{z}_1} e_{m,n} + \mathbf{Q}_{z_2}^\dagger \mathbf{Q}_{z_1} e_{m,n} \\ &= \sqrt{2(m+1)\hbar} \mathbf{Q}_{z_2} e_{m+1,n} + \sqrt{2m\hbar} \mathbf{Q}_{\bar{z}_2} e_{m-1,n} \\ &= 2\sqrt{(m+1)n\hbar} e_{m+1,n-1} + 2\sqrt{m(n+1)\hbar} e_{m-1,n+1} = \mathbf{Q}_{2\pi_1} e_{m,n}. \end{aligned}$$

Similarly, the linear operators \mathbf{Q}_{π_k} for $k=2, 3, 4$ are also self adjoint.

We now show that the operators \mathbf{Q}_{π_1} , \mathbf{Q}_{π_2} , and \mathbf{Q}_{π_3} satisfy the commutation relations

$$[\mathbf{Q}_{\pi_1}, \mathbf{Q}_{\pi_2}] = \frac{\hbar}{i}(2\mathbf{Q}_{\pi_3}), \quad [\mathbf{Q}_{\pi_1}, \mathbf{Q}_{\pi_3}] = -\frac{\hbar}{i}(2\mathbf{Q}_{\pi_2}), \quad [\mathbf{Q}_{\pi_2}, \mathbf{Q}_{\pi_3}] = \frac{\hbar}{i}(2\mathbf{Q}_{\pi_1}),$$

which define a Lie algebra that is isomorphic to $\mathfrak{su}(2)$.

We compute

$$\begin{aligned} [\mathbf{Q}_{\pi_1}, \mathbf{Q}_{\pi_2}] e_{m,n} &= \mathbf{Q}_{\pi_1} \mathbf{Q}_{\pi_2} e_{m,n} - \mathbf{Q}_{\pi_2} \mathbf{Q}_{\pi_1} e_{m,n} \\ &= \mathbf{Q}_{\pi_1} \frac{\hbar}{i} (\sqrt{m(n+1)} e_{m-1,n+1} - \sqrt{(m+1)n} e_{m+1,n-1}) \\ &\quad - \mathbf{Q}_{\pi_2} \hbar (\sqrt{m(n+1)} e_{m-1,n+1} + \sqrt{(m+1)n} e_{m+1,n-1}) \\ &= \frac{\hbar}{i} \sqrt{m(n+1)} \hbar (\sqrt{(m-1)(n+2)} e_{m-2,n+2} + \sqrt{m(n+1)} e_{m,n}) \\ &\quad - \frac{\hbar}{i} \sqrt{m(n+1)} \hbar (\sqrt{(m+1)n} e_{m,n} + \sqrt{(m+2)(n-1)} e_{m+2,n-2}) \\ &\quad - \hbar \sqrt{m(n+1)} \frac{\hbar}{i} (\sqrt{(m-1)(n+2)} e_{m-2,n+2} + \sqrt{m(n+1)} e_{m,n}) \\ &\quad - \hbar \sqrt{(m+1)n} \frac{\hbar}{i} (\sqrt{(m+1)n} e_{m,n} + \sqrt{(m+2)(n-1)} e_{m+2,n-2}) \\ &= \frac{\hbar}{i} (2\hbar(m-n) e_{m,n}) = \frac{\hbar}{i} (2\mathbf{Q}_{\pi_3}) e_{m,n}. \end{aligned}$$

Similarly $[\mathbf{Q}_{\pi_1}, \mathbf{Q}_{\pi_3}] e_{m,n} = -\frac{\hbar}{i} (2\mathbf{Q}_{\pi_2}) e_{m,n}$ and $[\mathbf{Q}_{\pi_2}, \mathbf{Q}_{\pi_3}] e_{m,n} = \frac{\hbar}{i} (2\mathbf{Q}_{\pi_1}) e_{m,n}$.

We can rewrite these commutation relations as

$$\begin{aligned} [\frac{1}{i\hbar} \mathbf{Q}_{\pi_1}, \frac{1}{i\hbar} \mathbf{Q}_{\pi_2}] &= \frac{1}{i\hbar} (2\mathbf{Q}_{\pi_3}), \quad [\frac{1}{i\hbar} \mathbf{Q}_{\pi_1}, \frac{1}{i\hbar} \mathbf{Q}_{\pi_3}] = -\frac{1}{i\hbar} (2\mathbf{Q}_{\pi_2}), \\ [\frac{1}{i\hbar} \mathbf{Q}_{\pi_2}, \frac{1}{i\hbar} \mathbf{Q}_{\pi_3}] &= \frac{1}{i\hbar} (2\mathbf{Q}_{\pi_1}). \end{aligned} \tag{32}$$

Thus we obtain the following representation of $\mathfrak{su}(2)$ on \mathfrak{H} by skew hermitian operators

$$\tilde{\mu} : \mathfrak{su}(2) \rightarrow \mathfrak{u}(\mathfrak{H}, \mathbb{C}) : \frac{1}{i\hbar} \mathbf{Q}_{\pi_j} \mapsto \tilde{\mu}(\frac{1}{i\hbar} \mathbf{Q}_{\pi_j}),$$

where $\tilde{\mu}(\frac{1}{i\hbar}\mathbf{Q}_{\pi_j})$ is the skew hermitian operator $\mathfrak{H} \rightarrow \mathfrak{H} : e_{m,n} \mapsto \frac{1}{i\hbar}\mathbf{Q}_{\pi_j}e_{m,n}$.

A straightforward computation shows that the quantum operator \mathbf{Q}_E commutes with the quantum operators \mathbf{Q}_{π_j} for $j = 1, 2, 3$. Thus in addition to the bracket relations (32) for $\frac{1}{i\hbar}\mathbf{Q}_{\pi_j}$, $j = 1, 2, 3$ we have the relations

$$[\frac{1}{i\hbar}\mathbf{Q}_E, \frac{1}{i\hbar}\mathbf{Q}_{\pi_j}] = 0, \quad \text{for } j = 1, 2, 3. \quad (33)$$

The bracket relations (32) and (33) define the Lie algebra $\mathfrak{u}(2)$.

Therefore we obtain the following representation of $\mathfrak{u}(2)$ on the Hilbert space $(\mathfrak{H}, (\cdot | \cdot))$ by linear skew hermitian maps

$$\mu : \mathfrak{u}(2) \rightarrow \mathfrak{u}(\mathfrak{H}, \mathbb{C}) : \begin{cases} \frac{1}{i\hbar}\mathbf{Q}_E, \\ \frac{1}{i\hbar}\mathbf{Q}_{\pi_j}, \quad j = 1, 2, 3 \end{cases} \mapsto \begin{cases} \frac{1}{i\hbar}\mu(\mathbf{Q}_E), \\ \frac{1}{i\hbar}\tilde{\mu}(\mathbf{Q}_{\pi_j}), \quad j = 1, 2, 3, \end{cases}$$

where $\mu(\frac{1}{i\hbar}\mathbf{Q}_E) : \mathfrak{H} \rightarrow \mathfrak{H} : e_{m,n} \mapsto \frac{1}{i\hbar}\mathbf{Q}_E e_{m,n}$.

Recall that $\mathfrak{H}_N = \text{span}_{\mathbb{C}}\{e_{mn} \in \mathfrak{H} \mid m+n = N\}$. Then \mathfrak{H}_N is a complex $(N+1)$ -dimensional subspace of \mathfrak{H} , which is invariant under \mathbf{Q}_E , being its eigenspace corresponding to the eigenvalue $N\hbar$. Therefore $\mathfrak{H} = \bigoplus_{N \geq 0} \mathfrak{H}_N$ is an orthogonal direct sum decomposition, because \mathbf{Q}_E is self adjoint. Since

$$\begin{aligned} \mathbf{Q}_{\pi_1} e_{m,n} &= \hbar(\sqrt{m(n+1)} e_{m-1,n+1} + \sqrt{(m+1)n} e_{m+1,n-1}) \\ \mathbf{Q}_{\pi_2} e_{m,n} &= \frac{\hbar}{i}(\sqrt{m(n+1)} e_{m-1,n+1} - \sqrt{(m+1)n} e_{m+1,n-1}) \\ \mathbf{Q}_{\pi_3} e_{m,n} &= \hbar(m-n)e_{m,n} \end{aligned}$$

it follows that for $j = 1, 2, 3$ the linear skew hermitian quantum operator $\frac{1}{i\hbar}\mathbf{Q}_{\pi_j}$ maps \mathfrak{H}_N into itself. Thus for $j = 1, 2, 3$ the linear skew hermitian operators $\frac{1}{i\hbar}\mathbf{Q}_{\pi_j}|_{\mathfrak{H}_N}$ define a representation of $\mathfrak{su}(2)$ on \mathfrak{H}_N , which is clearly irreducible. So we have decomposed the representation of $\mathfrak{u}(2)$ on \mathfrak{H} into a sum of irreducible $\mathfrak{u}(2)$ representations.

5 Quantization of the reduced system

For fixed $e \geq 0$ we quantize the classical system $(\tilde{\pi}_3, S_e^2, \omega_e)$ obtained by reducing the harmonic oscillator symmetry of the Hamiltonian system $(L, T^*\mathbb{R}^2, \omega)$, see §3. We want to understand the relation of its quantum spectrum to the quantum spectrum of the 2-dimensional harmonic oscillator.

Recall that (S_e^2, ω_e) is symplectomorphic to a coadjoint orbit of $SU(2)$. Moreover, functions $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3$ correspond to evaluation of elements of the orbit on a basis of $\mathfrak{su}(2)$, see appendix. Hence, quantization of the system $(\tilde{\pi}_3, S_e^2, \omega_e)$ should lead to a representation of $SU(2)$.

We assume that the symplectic manifold (S_e^2, ω_e) is prequantizable. In other words, for $q \in \mathbb{Z}$ we have

$$qh = \int_{S_e^2} \omega_e = \frac{1}{2e} \int_{S_e^2} \text{vol}_{S_e^2} = \frac{1}{2e} (4\pi e^2) = 2\pi e,$$

that is, $0 \leq e = q\hbar$, where $\hbar = \frac{h}{2\pi}$. Thus q is a fixed nonnegative integer.

The Bohr-Sommerfeld conditions for the integrable reduced system $(\tilde{\pi}_3, S_e^2, \omega_e)$ in action-angle coordinates (a, ψ) , see §3, are for $p \in \mathbb{Z}$

$$ph = \oint_{a^{-1}(\ell)} a \, d\psi = \ell \int_0^\pi 2 \, d\psi = 2\pi\ell.$$

The second equality above follows because the curve $[0, \pi] \rightarrow S_e^2 : \psi \mapsto (\sqrt{e^2 - \ell^2} \cos 2\psi, \sqrt{e^2 - \ell^2} \sin 2\psi, \ell)$ parametrizes $a^{-1}(\ell)$. Thus $\ell = p\hbar$. Since $\ell = \pi_3 \cos \theta_p$ and $|\cos \theta_p| \leq 1$, it follows that $|\ell| \leq e$, which implies $|p| \leq q$ and $\cos \theta_p = \frac{p}{q}$. So the Bohr-Sommerfeld set \tilde{S}_q for the reduced integrable system $(\pi_3|S_{qh}^2, S_{qh}^2, \omega_{qh})$ with $q \in \mathbb{Z}_{\geq 0}$ is

$$\tilde{S}_q = \{(q\hbar \sin \theta_p \cos 2\psi, q\hbar \sin \theta_p \sin 2\psi, p\hbar) \in S_{qh}^2 \mid p \in \mathbb{Z} \text{ with } |p| \leq q\}.$$

The set \tilde{S}_q is a disjoint union of $2q - 1$ circles of radius $(\sqrt{q^2 - p^2})\hbar$ when $|p| < q$ and two points $(0, 0, \pm q\hbar)$ when $p = \pm q$.

Let $\tilde{e}_{p,q}$ with $|p| \leq q$ be a vector in $\tilde{\mathfrak{H}}_q$ which corresponds to the 1-torus $\{(q\hbar \sin \theta_p \cos 2\psi, q\hbar \sin \theta_p \sin 2\psi, p\hbar) \in S_{qh}^2 \mid \psi \in [0, \pi]\}$ in the Bohr-Sommerfeld set \tilde{S}_q . Define an inner product $\langle \cdot, \cdot \rangle$ on $\tilde{\mathfrak{H}}_q$ so that $\langle \tilde{e}_{p,q} | \tilde{e}_{p',q'} \rangle = \delta_{(p,q),(p',q')}$. From now on we use the notation $\tilde{\pi}_j$, $j = 1, 2, 3$ for $\pi_j|S_{qh}^2$. For each integer p with $|p| \leq q$ we have

$$\mathbf{Q}_{\tilde{\pi}_3} \tilde{e}_{p,q} = q\hbar \cos \theta_p \tilde{e}_{p,q} = p\hbar \tilde{e}_{p,q}. \quad (34)$$

So $\tilde{\mathfrak{H}}_q = \text{span}\{\tilde{e}_{pq} \mid |p| \leq q\}$ consists of eigenvectors of $\mathbf{Q}_{\tilde{\pi}_3}$.

We want to define shifting operators $\tilde{\mathbf{a}}_q$ and $\tilde{\mathbf{a}}_q^\dagger$ on $\tilde{\mathfrak{H}}$ so that

$$\tilde{\mathbf{a}}_q \tilde{e}_{p,q} = \tilde{e}_{p-1,q} \quad \text{and} \quad \tilde{\mathbf{a}}_q^\dagger \tilde{e}_{p,q} = \tilde{e}_{p+1,q}$$

By general theory, we expect to identify the operators $\tilde{\mathbf{a}}_q$ and $\tilde{\mathbf{a}}_q^\dagger$ with the quantum operators $\mathbf{Q}_{e^{i\psi}}$ and $\mathbf{Q}_{e^{-i\psi}}$, respectively. However, the functions $e^{\pm i\psi}$ are not single-valued on the complement of the poles in $S_{q\hbar}^2$. In order to get single-valued functions, we have take squares of $e^{\pm i\psi}$, namely the functions $e^{\pm 2i\psi}$. The functions $e^{\pm 2i\psi}$ do not extend to smooth functions on $S_{q\hbar}^2$. However, the functions

$$\tilde{\pi}_\pm = (\sqrt{(q\hbar)^2 - \tilde{\pi}_3^2}) e^{\pm 2i\psi} = q\hbar \sin \theta \cos 2\psi \pm i q\hbar \sin \theta \sin 2\psi = \tilde{\pi}_1 \pm i\tilde{\pi}_2$$

do. Moreover, we have

$$\{\tilde{\pi}_\pm, \tilde{\pi}_3\} = \{\tilde{\pi}_1, \tilde{\pi}_3\} \pm i \{\tilde{\pi}_2, \tilde{\pi}_3\} = 2\tilde{\pi}_2 \pm 2i\tilde{\pi}_1 = \pm 2i(\tilde{\pi}_1 \pm i\tilde{\pi}_2) = \pm 2i \tilde{\pi}_\pm.$$

Under reduction the quantum operator $\frac{1}{i\hbar} \mathbf{Q}_E$ corresponds to the reduced quantum operator $\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_4}$ and the quantum operator $\frac{1}{i\hbar} \mathbf{Q}_L$ corresponds to the reduced quantum operator $\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_3}$.

As in the discussion of quantization of coadjoint in [4], we can define shifting operators $\mathbf{Q}_{\tilde{\pi}_\pm}$ as follows. Set

$$\mathbf{Q}_{\tilde{\pi}_+} \tilde{e}_{p,q} = b_p \tilde{e}_{p-2,q} \quad \text{and} \quad \mathbf{Q}_{\tilde{\pi}_+}^\dagger \tilde{e}_{p,q} = b_{p+2} \tilde{e}_{p+2,q}$$

and

$$\mathbf{Q}_{\tilde{\pi}_-} \tilde{e}_{p,q} = b_{p+2} \tilde{e}_{p+2,q} \quad \text{and} \quad \mathbf{Q}_{\tilde{\pi}_-}^\dagger \tilde{e}_{p,q} = b_p \tilde{e}_{p,q},$$

where $b_p \in \mathbb{R}$ and $b_{-q} = 0 = b_{q+2}$. Then

$$\mathbf{Q}_{\tilde{\pi}_+} \mathbf{Q}_{\tilde{\pi}_-} \tilde{e}_{p,q} = b_{p+2} \mathbf{Q}_{\tilde{\pi}_+} \tilde{e}_{p+2,q} = b_{p+2}^2 \tilde{e}_{p,q} \quad \text{and} \quad \mathbf{Q}_{\tilde{\pi}_-} \mathbf{Q}_{\tilde{\pi}_+} \tilde{e}_{p,q} = b_p^2 \tilde{e}_{p,q}.$$

So

$$[\mathbf{Q}_{\tilde{\pi}_+}, \mathbf{Q}_{\tilde{\pi}_-}] \tilde{e}_{p,q} = \mathbf{Q}_{\tilde{\pi}_+} \mathbf{Q}_{\tilde{\pi}_-} \tilde{e}_{p,q} - \mathbf{Q}_{\tilde{\pi}_-} \mathbf{Q}_{\tilde{\pi}_+} \tilde{e}_{p,q} = (b_{p+2}^2 - b_p^2) \tilde{e}_{p,q}.$$

From

$$\{\tilde{\pi}_+, \tilde{\pi}_-\} = \{\tilde{\pi}_1 + i\tilde{\pi}_2, \tilde{\pi}_1 - i\tilde{\pi}_2\} = -2i \{\tilde{\pi}_1, \tilde{\pi}_2\} = -4i \tilde{\pi}_3$$

it follows that we should have

$$[\mathbf{Q}_{\tilde{\pi}_+}, \mathbf{Q}_{\tilde{\pi}_-}] = -i\hbar \mathbf{Q}_{\{\tilde{\pi}_+, \tilde{\pi}_-\}} = -i\hbar \mathbf{Q}_{-4i \tilde{\pi}_3} = -4\hbar \mathbf{Q}_{\tilde{\pi}_3}. \quad (35)$$

Evaluating both sides of equation (35) on $\tilde{e}_{p,q}$ gives

$$b_{p+2}^2 - b_p^2 = -4\hbar^2 p,$$

for every $p = -q + 2k$, where $0 \leq k \leq q$ and $k \in \mathbb{Z}$. In particular we have $-4\hbar^2 q = b_{q+2}^2 - b_q^2 = -b_q^2$, since $b_{q+2} = 0$ and $-4\hbar^2(-q) = b_{-q+2}^2 - b_{-q}^2 = b_{-q+2}^2$, since $b_{-q} = 0$. Now for every $p = -q + 2k$, where $k \in \{0, 1, 2, \dots, q\}$ we have

$$\begin{aligned} b_p^2 &= (b_p^2 - b_{p-2}^2) + (b_{p-2}^2 - b_{p-4}^2) + \dots + (b_{-q+2}^2 - b_{-q}^2) + b_{-q}^2 \\ &= -4\hbar^2(p-2) - 4\hbar^2(p-4) + \dots - 4\hbar^2(-q), \quad \text{since } b_{-q} = 0 \\ &= -4\hbar^2 \sum_{\ell=1}^k (p-2\ell), \quad \text{since } p = -q + 2k \\ &= -4\hbar^2(pk - (k+1)k) = 4\hbar^2 k(q+1-k) \\ &= \hbar^2(p+q)(q-p+2). \end{aligned} \tag{36}$$

Therefore when $p \in \{-q, -q+2, -q+4, \dots, q-2, q\}$ and $2k = p+q$ we have

$$\begin{aligned} \mathbf{Q}_{\tilde{\pi}_+} \tilde{e}_{p,q} &= b_p \tilde{e}_{p-2,q} = 2\hbar \sqrt{k(q+1-k)} \tilde{e}_{p-2,q} \\ &= \hbar \sqrt{(p+q)(q-p+2)} \tilde{e}_{p-2,q} \end{aligned}$$

and

$$\begin{aligned} \mathbf{Q}_{\tilde{\pi}_-} \tilde{e}_{p,q} &= b_{p+2} \tilde{e}_{p+2,q} = 2\hbar \sqrt{(k+1)(q-k)} \tilde{e}_{p+2,q} \\ &= \hbar \sqrt{(p+q+2)(q-p)} \tilde{e}_{p+2,q} \end{aligned}$$

We now show that equation (35) holds. We compute

$$\begin{aligned} (\mathbf{Q}_{\tilde{\pi}_+} \mathbf{Q}_{\tilde{\pi}_-} - \mathbf{Q}_{\tilde{\pi}_-} \mathbf{Q}_{\tilde{\pi}_+}) \tilde{e}_{p,q} &= \mathbf{Q}_{\tilde{\pi}_+} b_{p+2} \tilde{e}_{p+2,q} - \mathbf{Q}_{\tilde{\pi}_-} b_p \tilde{e}_{p-2,q} \\ &= (b_{p+2}^2 - b_p^2) \tilde{e}_{p,q} = 4\hbar^2 ((k+1)(q-k) - k(q+1-k)) \tilde{e}_{p,q} \\ &= -4\hbar^2 (-q+2k) \tilde{e}_{p,q} = -4\hbar^2 p \tilde{e}_{p,q} = -4\hbar \mathbf{Q}_{\tilde{\pi}_3} \tilde{e}_{p,q}. \end{aligned}$$

Since $\tilde{\pi}_1 = \frac{1}{2}(\tilde{\pi}_+ + \tilde{\pi}_-)$ and $\tilde{\pi}_2 = \frac{1}{2i}(\tilde{\pi}_+ - \tilde{\pi}_-)$, we get

$$\mathbf{Q}_{\tilde{\pi}_1} \tilde{e}_{p,q} = \frac{1}{2} \mathbf{Q}_{\tilde{\pi}_+} \tilde{e}_{p,q} + \frac{1}{2} \mathbf{Q}_{\tilde{\pi}_-} \tilde{e}_{p,q} = \frac{1}{2} b_p \tilde{e}_{p-2,q} + \frac{1}{2} b_{p+2} \tilde{e}_{p+2,q}$$

and similarly

$$\mathbf{Q}_{\tilde{\pi}_2} \tilde{e}_{p,q} = \frac{1}{2i} b_p \tilde{e}_{p-2,q} - \frac{1}{2i} b_{p+2} \tilde{e}_{p+2,q}. \tag{37}$$

The following calculation shows that the linear operator $\mathbf{Q}_{\tilde{\pi}_1}$ is self adjoint on $\tilde{\mathfrak{H}}_q$.

$$\mathbf{Q}_{\tilde{\pi}_1}^\dagger \tilde{e}_{p,q} = \mathbf{Q}_{\tilde{\pi}_+}^\dagger \tilde{e}_{p,q} + \mathbf{Q}_{\tilde{\pi}_-}^\dagger \tilde{e}_{p,q} = b_{p+2} \tilde{e}_{p+2,q} + b_p \tilde{e}_{p-2,q} = \mathbf{Q}_{\tilde{\pi}_1} \tilde{e}_{p,q}.$$

Similarly, the operators $\mathbf{Q}_{\tilde{\pi}_k}$ for $k = 2, 3, 4$ are self adjoint.

The operators $\mathbf{Q}_{\tilde{\pi}_1}$, $\mathbf{Q}_{\tilde{\pi}_2}$, and $\mathbf{Q}_{\tilde{\pi}_3}$ satisfy the commutation relations

$$[\mathbf{Q}_{\tilde{\pi}_1}, \mathbf{Q}_{\tilde{\pi}_2}] = -i\hbar(2\mathbf{Q}_{\tilde{\pi}_3}), \quad [\mathbf{Q}_{\tilde{\pi}_2}, \mathbf{Q}_{\tilde{\pi}_3}] = -i\hbar(2\mathbf{Q}_{\tilde{\pi}_1}), \quad [\mathbf{Q}_{\tilde{\pi}_3}, \mathbf{Q}_{\tilde{\pi}_1}] = i\hbar(-2\mathbf{Q}_{\tilde{\pi}_2}),$$

because

$$\begin{aligned} [\mathbf{Q}_{\tilde{\pi}_1}, \mathbf{Q}_{\tilde{\pi}_2}] \tilde{e}_{p,q} &= \mathbf{Q}_{\tilde{\pi}_1} \mathbf{Q}_{\tilde{\pi}_2} \tilde{e}_{p,q} - \mathbf{Q}_{\tilde{\pi}_2} \mathbf{Q}_{\tilde{\pi}_1} \tilde{e}_{p,q} \\ &= \frac{1}{2i} b_p \mathbf{Q}_{\tilde{\pi}_1} \tilde{e}_{p-2,q} - \frac{1}{2i} b_{p+2} \mathbf{Q}_{\tilde{\pi}_1} \tilde{e}_{p+2,q} \\ &\quad - \frac{1}{2} b_p \mathbf{Q}_{\tilde{\pi}_2} \tilde{e}_{p-2,q} - \frac{1}{2} b_{p+2} \mathbf{Q}_{\tilde{\pi}_2} \tilde{e}_{p+2,q} \\ &= \frac{1}{2i} b_p \left(\frac{1}{2} b_{\tilde{m}-2} \tilde{e}_{p-2,q} + \frac{1}{2} b_p \tilde{e}_{p,q} \right) \\ &\quad - \frac{1}{2i} b_{p+2} \left(\frac{1}{2} b_{p+2} \tilde{e}_{p+2,q} + \frac{1}{2} b_{p+4} \tilde{e}_{p+4,q} \right) \\ &\quad - \frac{1}{2} b_p \left(\frac{1}{2i} b_{p-2} \tilde{e}_{p-2,q} - \frac{1}{2i} b_p \tilde{e}_{p,q} \right) \\ &\quad - \frac{1}{2} b_{p+2} \left(\frac{1}{2i} b_{p+2} \tilde{e}_{p-2,q} - \frac{1}{2i} b_{p+4} \tilde{e}_{p,q} \right) \\ &= \frac{1}{4i} (2b_p^2 - 2b_{p+2}^2) \tilde{e}_{p,q} = -i\hbar(2p\hbar) \tilde{e}_{p,q} \\ &= -i\hbar(2\mathbf{Q}_{\tilde{\pi}_3}) \tilde{e}_{p,q}. \end{aligned}$$

Similarly,

$$[\mathbf{Q}_{\tilde{\pi}_2}, \mathbf{Q}_{\tilde{\pi}_3}] \tilde{e}_{p,q} = -i\hbar(2\mathbf{Q}_{\tilde{\pi}_1}) \tilde{e}_{p,q} \quad \text{and} \quad [\mathbf{Q}_{\tilde{\pi}_1}, \mathbf{Q}_{\tilde{\pi}_3}] \tilde{e}_{m,q} = -i\hbar(-2\mathbf{Q}_{\tilde{\pi}_2}) \tilde{e}_{p,q}.$$

Therefore the skew hermitian operators $\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_1}$, $\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_2}$, and $\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_3}$ satisfy the bracket relations

$$[\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_1}, \frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_2}] = \frac{1}{i\hbar} (2\mathbf{Q}_{\tilde{\pi}_3}), \quad [\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_1}, \frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_3}] = \frac{1}{i\hbar} (-2\mathbf{Q}_{\tilde{\pi}_2}), \quad [\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_2}, \frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_3}] = \frac{1}{i\hbar} (2\mathbf{Q}_{\tilde{\pi}_1}),$$

which defines the Lie algebra $\mathfrak{su}(2)$. Compare with table 1.

Next we construct a representation of $\mathfrak{su}(2)$ by skew hermitian linear operators. The bracket relations show that for $j = 1, 2, 3$ the mapping

$$\tilde{\mu}_{q+1} : \mathfrak{su}(2) \rightarrow \mathfrak{u}(\tilde{\mathfrak{H}}_q, \mathbb{C}) : \frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_j} \rightarrow \tilde{\mu}_{q+1}(\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_j}),$$

where $\tilde{\mu}_{q+1}(\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_j}) : \tilde{\mathfrak{H}}_q \rightarrow \tilde{\mathfrak{H}}_q : \tilde{e}_{p,q} \mapsto \frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_j} \tilde{e}_{p,q}$ is a faithful complex $2q + 1$ -dimensional representation of $\mathfrak{su}(2)$ on $\tilde{\mathfrak{H}}_q$ by skew hermitian linear operators.

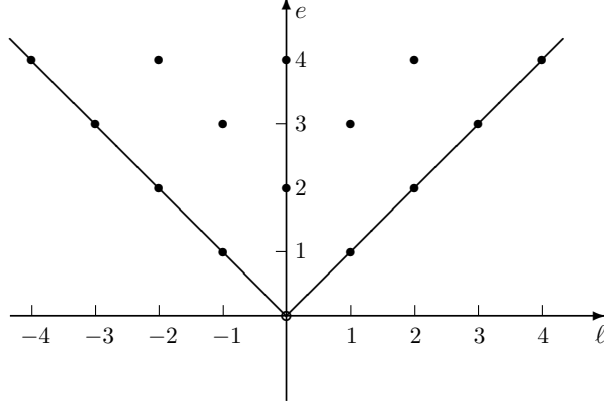


Figure 3. The quantum lattice for the 2-dimensional harmonic oscillator after reducing the oscillator symmetry. The axes are in units of \hbar .

This representation is can be decomposed into a sum of two irreducible representations as follows. Let

$$\tilde{\mathfrak{H}}_q^0 = \bigoplus_{m=0}^n \text{span}\{(Q_{\tilde{\pi}_+})^m \tilde{e}_{q,q}\} = \bigoplus_{m=0}^n \text{span}\{\tilde{e}_{q-2p,q}\}$$

and

$$\tilde{\mathfrak{H}}_q^1 = \bigoplus_{m=0}^{n-1} \text{span}\{(Q_{\tilde{\pi}_+})^m \tilde{e}_{q-1,q}\} = \bigoplus_{m=0}^{n-1} \text{span}\{\tilde{e}_{q-1-2p,q}\}.$$

Then, $\tilde{\mathfrak{H}}_q^0$ and $\tilde{\mathfrak{H}}_q^1$ are irreducible representations of $\mathfrak{su}(2)$ of dimension $q+1$ and q , respectively. Moreover, $\tilde{\mathfrak{H}}_q = \tilde{\mathfrak{H}}_q^0 \oplus \tilde{\mathfrak{H}}_q^1$. Thus, Bohr-Sommerfeld-Heisenberg quantization of coadjoint orbits of $\text{SU}(2)$ leads to a *reducible* $\mathfrak{su}(2)$ representation by skew hermitian operators.

Using the Campbell–Baker–Hausdorff formula and the fact that $\text{SU}(2)$ is compact, we can exponentiate the representation $\tilde{\mu}_{q+1}$ to a unitary representation R_q of $\text{SU}(2)$ on $\tilde{\mathfrak{H}}_q$. For more details see [4].

6 Quantum reduction

We now discuss the explicit relation between the Bohr-Sommerfeld quantization of the 2-dimensional harmonic oscillator and the Bohr-Sommerfeld quantization of the reduced systems. Recall that the Bohr-Sommerfeld quantization of the 2-dimensional harmonic oscillator gives rise to the skew hermitian quantum operators

$$\begin{aligned} \mathbf{Q}_{\pi_1} e_{m,n} &= \hbar (\sqrt{m(n+1)} e_{m-1,n+1} + \sqrt{(m+1)n} e_{m+1,n-1}) \\ \mathbf{Q}_{\pi_2} e_{m,n} &= \frac{\hbar}{i} (\sqrt{m(n+1)} e_{m-1,n+1} - \sqrt{(m+1)n} e_{m+1,n-1}) \\ \mathbf{Q}_{\pi_3} e_{mn} &= \mathbf{Q}_L e_{mn} = (m-n) \hbar e_{m,n} \\ \mathbf{Q}_{\pi_4} e_{mn} &= \mathbf{Q}_E e_{m,n} = (m+n) \hbar e_{m,n} \end{aligned} \tag{38}$$

$$\tag{39}$$

on the Hilbert space $\mathfrak{H} = \{e_{m,n} \mid m \geq 0 \& n \geq 0\}$ with inner product $\langle \cdot, \cdot \rangle$. Operators, \mathbf{Q}_{π_1} , \mathbf{Q}_{π_2} and \mathbf{Q}_{π_3} generate a representation of $\mathfrak{su}(2)$, which integrates to a unitary representation of $\mathrm{SU}(2)$ on \mathfrak{H} . We have seen that \mathbf{Q}_{π_1} , \mathbf{Q}_{π_2} and \mathbf{Q}_{π_3} commute with $\mathbf{Q}_{\pi_4} = \mathbf{Q}_E$. Hence, fixing an eigenvalue $e = q\hbar$ of \mathbf{Q}_E , for some $q = 0, 1, \dots$, we get an invariant subspace \mathfrak{H}_q of \mathfrak{H} carrying a unitary representation \mathbf{U}_q of $\mathrm{SU}(2)$. It follows from the equations above that $q = m + n$, for $m \geq 0$, $n \geq 0$. Hence,

$$\mathfrak{H}_q = \mathrm{span}\{e_{m,n} \mid m + n = q, m \geq 0 \& n \geq 0\} = \bigoplus_{m=0}^q \mathrm{span}\{e_{m,q-m}\}$$

has dimension $q + 1$. We shall show that the representation \mathbf{U}_q on \mathfrak{H}_q is irreducible. Consider the quantization of the reduced symplectic manifold (S_e^2, ω_e) corresponding to $e = q\hbar$. We get the quantum operators

$$\begin{aligned} \mathbf{Q}_{\tilde{\pi}_1} \tilde{e}_{p,q} &= \frac{1}{2} \hbar (\sqrt{(p+q)(q-p+2)} \tilde{e}_{p-2,q} + \sqrt{(p+q+2)(q-p)} \tilde{e}_{p+2,q}) \\ \mathbf{Q}_{\tilde{\pi}_2} \tilde{e}_{p,q} &= \frac{1}{2i} \hbar (\sqrt{(p+q)(q-p+2)} \tilde{e}_{p-2,q} - \sqrt{(p+q+2)(q-p)} \tilde{e}_{p+2,q}) \\ \mathbf{Q}_{\tilde{\pi}_3} \tilde{e}_{p,q} &= p \hbar \tilde{e}_{p,q} \end{aligned}$$

on the Hilbert space $\tilde{\mathfrak{H}}_q = \{\tilde{e}_{p,q} \mid |p| \leq q\}$ with inner product $\langle \cdot, \cdot \rangle$.

To each $e_{m,n} \in \mathfrak{H}_q$, we associate $\tilde{e}_{m-n,m+n} \in \tilde{\mathfrak{H}}$. By definition of \mathfrak{H}_q , we have $m + n = q$. Since $-(m+n) \leq m-n \leq m+n$, it follows that

$|m - n| \leq m + n = q$. Therefore, $\tilde{e}_{m-n, m+n} = \tilde{e}_{m-n, q} \in \tilde{\mathfrak{H}}_q$. This yields a linear map from $\mathcal{I}_q : \mathfrak{H}_q \rightarrow \tilde{\mathfrak{H}}_q$. The range of this map is

$$\mathcal{I}_q \left(\bigoplus_{m=0}^q \text{span}\{e_{m, q-m}\} \right) = \bigoplus_{m=0}^q \text{span}\{\mathcal{I}_q e_{m, q-m}\} = \bigoplus_{m=0}^q \text{span}\{\tilde{e}_{2m-q, q}\} = \tilde{\mathfrak{H}}_q^0.$$

Let $\mathcal{I}_q^0 : \mathfrak{H}_q \rightarrow \tilde{\mathfrak{H}}_q^0$ the restriction of \mathcal{I}_q to codomain $\tilde{\mathfrak{H}}_q^0$. The map \mathcal{I}_q^0 is bijective, because its inverse is given by $\tilde{e}_{p, q} \mapsto e_{\frac{1}{2}(p+q), \frac{1}{2}(q-p)}$. Now

$$(\mathbf{Q}_{\tilde{\pi}_4} \circ \mathcal{I}_q^0) e_{m, n} = \mathbf{Q}_{\tilde{\pi}_4} \tilde{e}_{m-n, q} = q \hbar \tilde{e}_{m-n, q} = (m+n) \hbar \mathcal{I}_q^0 e_{m, n} = (\mathcal{I}_q^0 \circ \mathbf{Q}_E) e_{m, n}$$

and similarly $(\mathbf{Q}_{\tilde{\pi}_3} \circ \mathcal{I}_q^0) e_{m, n} = (\mathcal{I}_q^0 \circ \mathbf{Q}_L) e_{m, n}$. Also

$$\begin{aligned} (\mathbf{Q}_{\tilde{\pi}_1} \circ \mathcal{I}_q^0) e_{m, n} &= \mathbf{Q}_{\tilde{\pi}_1} \tilde{e}_{p, q} \\ &= \frac{1}{2} \hbar \mathcal{I}_q^0 (\sqrt{2m(2n+2)} e_{m-1, n+1} + \sqrt{(2m+2)2n} e_{m+1, n-1}) \\ &= (\mathcal{I}_q^0 \circ \mathbf{Q}_{\pi_1}) e_{m, n} \end{aligned}$$

and similarly $(\mathbf{Q}_{\tilde{\pi}_2} \circ \mathcal{I}_q^0) e_{m, n} = (\mathcal{I}_q^0 \circ \mathbf{Q}_{\pi_2}) e_{m, n}$. Note that the quantum spectrum of the quantized reduced system on the horizontal line $e = q$ in figure 3 corresponds to the diagonal line $e = m + n$ in figure 1 of the quantum spectrum of the quantized 2-dimensional harmonic oscillator.

The operator $\mathcal{I}_q^0 : \mathfrak{H}_q \rightarrow \tilde{\mathfrak{H}}_q^0$ conjugates the representation of $\mathfrak{su}(2)$ generated by the skew hermitian operators $\{\frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_1}, \frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_2}, \frac{1}{i\hbar} \mathbf{Q}_{\tilde{\pi}_3}\}$ on $\tilde{\mathfrak{H}}_q^0$ and the representation on \mathfrak{H}_q generated by the skew hermitian operators $\{\frac{1}{i\hbar} \mathbf{Q}_{\pi_1}, \frac{1}{i\hbar} \mathbf{Q}_{\pi_2}, \frac{1}{i\hbar} \mathbf{Q}_L\}$. Since, the representation of $\mathfrak{su}(2)$ on $\tilde{\mathfrak{H}}_q^0$ is irreducible, it follows that the representation of $\mathfrak{su}(2)$ on \mathfrak{H}_q is irreducible.

We can extend the representations of $\mathfrak{su}(2)$ discussed here, to a representation of $\mathfrak{u}(2)$ by adjoining the operator $\frac{1}{i\hbar} \mathbf{Q}_E$, which commutes with $\frac{1}{i\hbar} \mathbf{Q}_{\pi_1}$, $\frac{1}{i\hbar} \mathbf{Q}_{\pi_2}$, and $\frac{1}{i\hbar} \mathbf{Q}_L$. We can decompose the Hilbert space of our quantization of the harmonic oscillator as follows $\mathfrak{H} = \bigoplus_{q \geq 0} \mathfrak{H}_q$, where $\mathfrak{H}_q = \{e_{m, n} \in \mathfrak{H} \mid m + n = q\}$. Let $\tilde{\mathfrak{H}}^0 = \bigoplus_{q \geq 0} \tilde{\mathfrak{H}}_q^0$ and $\mathcal{I}^0 : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}^0 : e_{m, n} \mapsto \mathcal{I}_{m+n}^0 e_{m, n}$. The preceding discussion gives a decomposition of the unitary representation of $\mathfrak{U}(2)$, obtained by Bohr-Sommerfeld-Heisenberg quantization of the harmonic oscillator, into irreducible unitary representations of $\mathfrak{U}(2)$, each of which occurs with multiplicity 1.

We have shown that in 2-dimensional harmonic oscillator, each irreducible component of the quantization representation of $U(2)$ can be identified from the quantization of the corresponding coadjoint orbit reduced system using the inverse image of the momentum map $J : \mathbb{R}^4 \rightarrow \mathfrak{su}(2)^*$. Therefore, one could say that the Bohr-Sommerfeld-Heisenberg quantization of harmonic oscillator commutes with reduction.² However, we have to remember that this notion of commutation of quantization and reduction is weaker than that used by Guillemin and Sternberg [6] because in our case the quantization of the reduced space has a component $\tilde{\mathfrak{H}}_q^1$, which does not occur in \mathfrak{H}_q .

7 Appendix. Facts about $SU(2)$ coadjoint orbits

The linear action of $SU(2) = \{U = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \in GL(2, \mathbb{C}) \mid |\alpha|^2 + |\beta|^2 = 1\}$ on \mathbb{C}^2 with coordinates $z = (z_1, z_2)$ given by $SU(2) \times \mathbb{C}^2 \rightarrow \mathbb{C}^2 : (U, z) \mapsto Uz$ has the following properties.

- 1) It preserves the 2-form $\Omega = -\frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$.
- 2) It is Hamiltonian, because for every $u = \begin{pmatrix} iz & -y+ix \\ y+ix & -iz \end{pmatrix} \in \mathfrak{su}(2) = \{u \in \mathfrak{gl}(2, \mathbb{C}) \mid \bar{u}^T + u = 0\}$ the vector field

$$X^u = ((iz)z_1 + (-y+ix)z_2) \frac{\partial}{\partial z_1} + ((y-ix)z_1 + (-iz)z_2) \frac{\partial}{\partial z_2}$$

satisfies

$$\begin{aligned} X^u \lrcorner \Omega &= -\frac{i}{2} \left(((iz)z_1 + (-y+ix)z_2) d\bar{z}_1 + ((y-ix)z_1 + (-iz)z_2) d\bar{z}_2 \right) \\ &= \frac{1}{2} \bar{\partial} (x(z_1\bar{z}_2 + z_2\bar{z}_1) - iy(z_1\bar{z}_2 - z_2\bar{z}_2) + z(z_1\bar{z}_1 - z_2\bar{z}_2)) \\ &= \bar{\partial} J^u, \end{aligned}$$

where J^u is the Hermitian form

$$\begin{aligned} J^u(z) &= \frac{1}{2} \bar{z}^T (-iu) z \\ &= \frac{1}{2} (x(z_1\bar{z}_2 + z_2\bar{z}_1) - iy(z_1\bar{z}_2 - z_2\bar{z}_2) + z(z_1\bar{z}_1 - z_2\bar{z}_2)). \end{aligned} \quad (40)$$

From (40) it follows that the mapping $\mathfrak{su}(2) \rightarrow C^\infty(\mathbb{C}^2) : u \mapsto J^u$ is linear.

²Various interpretations of the term “quantization commutes with reduction” are discussed in [8].

3) It has an $\text{SU}(2)$ -momentum mapping

$$J : \mathbb{C}^2 \rightarrow \mathfrak{su}(2)^* : z \mapsto J(z),$$

where $J(z)u = J^u(z)$ for every $u \in \mathfrak{su}(2)$. The map J is $\text{SU}(2)$ -coadjoint equivariant, because for every $U \in \text{SU}(2)$ we have

$$\begin{aligned} J(Uz)u &= J^u(Uz) = -\frac{i}{2}((\overline{Uz})^T u(Uz)) = -\frac{i}{2}\bar{z}^T (\overline{U}^T u U)z \\ &= -\frac{i}{2}\bar{z}^T (\text{Ad}_{U^{-1}} u)z = J^{\text{Ad}_{U^{-1}} u}(z) = (\text{Ad}_{U^{-1}})^T(J(z))u. \end{aligned}$$

Identifying the basis

$$E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

of $\mathfrak{su}(2)$ with the standard basis e_1, e_2 , and e_3 , respectively, of \mathbb{R}^3 , the $\text{SU}(2)$ -momentum mapping J becomes the map

$$\tilde{\rho} : \mathbb{C}^2 \rightarrow \mathbb{R}^3 : (z_1, z_2) \mapsto \begin{pmatrix} \text{Re } z_1 \bar{z}_2 \\ \text{Im } z_1 \bar{z}_2 \\ \frac{1}{2}(|z_1|^2 - |z_2|^2) \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}.$$

In particular $\pi_j(z) = J(z)E_j$. Identifying \mathbb{C}^2 with coordinates $(z_1, z_2) = (\xi_1 + i\eta_1, \xi_2 + i\eta_2)$ with \mathbb{R}^4 with coordinates $(\xi_1, \xi_2, \eta_1, \eta_2)$ and then restricting to $S^3_{\sqrt{2e}} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 2e > 0\}$, the map $\tilde{\rho}$ becomes the Hopf fibration

$$\rho : S^3_{\sqrt{2e}} \subseteq \mathbb{R}^4 \rightarrow S^2_e \subseteq \mathbb{R}^3 : (\xi_1, \xi_2, \eta_1, \eta_2) \mapsto \begin{pmatrix} \xi_1 \xi_2 + \eta_1 \eta_2 \\ \xi_1 \eta_2 - \xi_2 \eta_1 \\ \frac{1}{2}((\xi_1^2 + \eta_1^2) - (\xi_2^2 + \eta_2^2)) \end{pmatrix},$$

which is the reduction map of the harmonic oscillator symmetry. Because $\text{SU}(2)$ acts transitively on $S^3_{\sqrt{2e}}$ and J is $\text{SU}(2)$ -coadjoint equivariant, the image S^2_e of the map ρ is an $\text{SU}(2)$ -coadjoint orbit $\mathcal{O}_{J(z_0)}$ through $J(z)$. Therefore for every $x \in S^2_e$ $\tilde{\pi}_j(x) = (\pi_j|_{\mathcal{O}_{J(z)}})(x) = (\text{Ad}_{U^{-1}}^T(J(z)))E_j$ for some $U \in \text{SU}(2)$ such that $x = \text{Ad}_{U^{-1}}^T(J(z))$.

4) It preserves the Hamiltonian $E = \frac{1}{2}(z_1 \bar{z}_1 + z_2 \bar{z}_2)$, since for every $U \in \text{SU}(2)$ we have

$$E(Uz) = \frac{1}{2}\bar{z}^T \overline{U}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Uz = \frac{1}{2}\bar{z}^T z = E(z).$$

The Hamiltonian vector field $X_E = X_1 \frac{\partial}{\partial z_1} + X_2 \frac{\partial}{\partial z_2}$ of E satisfies $X_E \lrcorner \Omega = \bar{\partial}E$, that is,

$$-\frac{i}{2}X_1 d\bar{z}_1 - \frac{i}{2}X_2 d\bar{z}_2 = \frac{1}{2}z_1 d\bar{z}_1 + \frac{1}{2}z_2 d\bar{z}_2.$$

So $X_E(z) = iz$. The flow of X_E defines the S^1 action $S^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2 : (t, z) \mapsto e^{it}z$, which commutes with the linear $SU(2)$ action on \mathbb{C}^2 .

The next discussion, analogous to the one used in [4], leads to an explicit expression for the standard symplectic form on the coadjoint orbit S_e^2 .

Define the bijective \mathbb{R} -linear mapping

$$j : \mathfrak{su}(2) \rightarrow \mathbb{R}^3 : u = \begin{pmatrix} iz & -y + ix \\ y + ix & -iz \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (41)$$

which identifies $\mathfrak{su}(2)$ with \mathbb{R}^3 . For $u, u' \in \mathfrak{su}(2)$ a straightforward calculation shows that

$$\begin{aligned} \left[\begin{pmatrix} iz & -y + ix \\ y + ix & -iz \end{pmatrix}, \begin{pmatrix} iz' & -y' + ix' \\ y' + ix' & -iz' \end{pmatrix} \right] &= \\ &= 2 \begin{pmatrix} i(xy' - yx') & -(zx' - xz') + i(yz' - zy') \\ (zx' - xz') + i(yz' - zy') & -i(xy' - yx') \end{pmatrix}. \end{aligned}$$

In other words,

$$j([u, u']) = 2j(u) \times j(u'), \quad \text{for every } u, u' \in \mathfrak{su}(2). \quad (42)$$

Using the bijective \mathbb{R} -linear map $i : \mathfrak{so}(3) \rightarrow \mathbb{R}^3 : \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to identify $\mathfrak{so}(3)$ with \mathbb{R}^3 , we may rewrite (42) as

$$j(\text{ad}_u u') = 2j(u) \times j(u'^{-1}(j(u))j(u')),$$

that is, $j(\text{ad}_u)j^{-1}(j(u'^{-1}(j(u))j(u'))$, or equivalently, as

$$j(\text{ad}_u)j^{-1} = 2i^{-1}(j(u)), \quad \text{for every } u \in \mathfrak{su}(2). \quad (43)$$

Exponentiating (43) gives

$$e^{2i^{-1}(j(u))} = e^{j(\text{ad}_u)j^{-1}} = j(e^{\text{ad}_u})j^{-1} = j(\text{Ad}_{\exp u})j^{-1}.$$

So

$$j(\text{Ad}_{\exp u})v = e^{2i^{-1}(j(u))}j(v), \quad \text{for every } u, v \in \mathfrak{su}(2). \quad (44)$$

The Killing form on $\mathfrak{su}(2)$ is $k(u, u') = \frac{1}{2} \operatorname{tr} u(\bar{u})^T$. Explicitly,

$$\begin{aligned} k(u, u') &= \frac{1}{2} \operatorname{tr} \begin{pmatrix} iz & -y+ix \\ y+ix & -iz \end{pmatrix} \begin{pmatrix} -iz & y-ix \\ -y-ix & iz \end{pmatrix} \\ &= \frac{1}{2} \operatorname{tr} \begin{pmatrix} zz' + xx' + yy' + iyy' - ixy' & * \\ * & zz' + xx' + yy'^* - iyx' + ixy' \end{pmatrix} \\ &= xx' + yy' + zz' = (j(u), j(u')), \end{aligned}$$

where (\cdot, \cdot) is the Euclidean inner product on \mathbb{R}^3 . In other words, $k = j^*(\cdot, \cdot)$. The Killing form is Ad-invariant, because if $U \in \operatorname{SU}(2)$, then

$$\begin{aligned} k(\operatorname{Ad}_U u, \operatorname{Ad}_U u') &= \frac{1}{2} \operatorname{tr} (\operatorname{Ad}_U u) (\overline{\operatorname{Ad}_U u'})^T = \frac{1}{2} \operatorname{tr} ((UuU^{-1})((\bar{U}^{-1})^T(\bar{u}')^T(\bar{U})^T)) \\ &= \frac{1}{2} \operatorname{tr} (Uu(\bar{u}')^T U^{-1}) = k(u, u'). \end{aligned}$$

Define the linear mapping $k^\flat : \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)^*$ by $k^\flat(u)v = k(u, v)$ for every $u, v \in \mathfrak{su}(2)$. Since k is nondegenerate, the map k^\flat is invertible with inverse k^\sharp . The linear map k^\sharp intertwines the coadjoint action of $\operatorname{SU}(2)$ on $\mathfrak{su}(2)^*$ with the adjoint action of $\operatorname{SU}(2)$ on $\mathfrak{su}(2)$. In other words, for every $u \in \mathfrak{su}(2)$ and every $U \in \operatorname{SU}(2)$ we have $k^\sharp(\operatorname{Ad}_{U^{-1}}^T k^\flat(u)) = \operatorname{Ad}_U u$. To see this we argue as follows. For every $v \in \mathfrak{su}(2)$ we have

$$\begin{aligned} (\operatorname{Ad}_{U^{-1}}^T k^\flat(u))v &= k^\flat(u) \operatorname{Ad}_{U^{-1}} v = k(u, \operatorname{Ad}_{U^{-1}} v) \\ &= k(\operatorname{Ad}_U u, v), \quad \text{since } k \text{ is Ad-invariant} \\ &= k^\flat(\operatorname{Ad}_U u)v, \end{aligned}$$

which proves the assertion. Thus we may identify $\operatorname{SU}(2)$ coadjoint orbits on $\mathfrak{su}(2)^*$ with $\operatorname{SU}(2)$ adjoint orbits on $\mathfrak{su}(2)$. Let \mathcal{O}_w be the $\operatorname{SU}(2)$ adjoint orbit through $w \in \mathfrak{su}(2)$. Using (44) we see that $j(\mathcal{O}_w) \subseteq S_e^2$, where $e^2 = k(w, w) > 0$. From (44) it follows that $j|_{\mathcal{O}_w}$ is locally an open mapping. Because S_e^2 is compact, we obtain $j(\mathcal{O}_w) = S_e^2$.

We now determine the standard symplectic form on S_e^2 . From [4] it follows that the standard symplectic form ω_w on \mathcal{O}_w is

$$\omega_w(v)(-\operatorname{ad}_v u, -\operatorname{ad}_v u') = -k(v, [u, u']), \quad \text{where } v \in \mathcal{O}_w \text{ and } u, u' \in \mathfrak{su}(2).$$

So $-\operatorname{ad}_v u, -\operatorname{ad}_v u' \in T_v \mathcal{O}_w$. Let $\omega_e = j^* \omega_w$. Then ω_e is a symplectic form on S_e^2 where $e^2 = k(w, w) > 0$. We now find an explicit expression for ω_e . Let $x = j(v)$, $y = j(u)$, and $y' = j(u')$. Then $x \in S_e^2$ and $x \times y, x \times y' \in T_x S_e^2$ and

$$\omega_e(x)(x \times y, x \times y') = \frac{1}{4} \omega_e(j(v))(2j(v) \times j(u), 2j(v) \times j(u'))$$

$$\begin{aligned}
&= \frac{1}{4} \omega_e(j(v))(j(\text{ad}_v u'), j(\text{ad}_v u')) \\
&= \frac{1}{4} \omega_w(v)(-\text{ad}_v u', -\text{ad}_v u'), \quad \text{since } \omega_e = j^* \omega_w \text{ and } j \text{ is linear} \\
&= -\frac{1}{4} k(v, [u, u']) = -\frac{1}{4} (j(v), 2j(u) \times j(u')) \\
&= -\frac{1}{2} (x, y \times y') = \frac{1}{2e} \text{vol}_{S_e^2}.
\end{aligned}$$

The last equality follows using spherical coordinates on S_e^2 .

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